1 The ADM Decomposition

Consider a \(d\)-dimensional spacetime with metric \(h_{ab}\). We start by identifying a scalar field \(t\) whose isosurfaces \(\Sigma_t\) are normal to the timelike unit vector given by

\[
u_a = -\alpha \partial_a t,
\]

where the lapse function \(\alpha\) is

\[
\alpha := \frac{1}{\sqrt{-h_{ab} \partial_a t \partial_b t}}.
\]

An observer whose worldline is tangent to \(\nu_a\) experiences an acceleration given by the vector

\[
\dot{a}_b = u^c \cdot d\nabla_c u_b,
\]

which is orthogonal to \(\nu_a\). The (spatial) metric on the \(d-1\) dimensional surface \(\Sigma_t\) is given by

\[
\sigma_{ab} = h_{ab} + \nu_a \nu_b.
\]

The intrinsic Ricci tensor built from this metric is denoted by \(R_{ab}\), and its Ricci scalar is \(R\). The covariant derivative on \(\Sigma_t\) is defined in terms of the \(d\) dimensional covariant derivative as

\[
D_a V_b := \sigma_{ac} \sigma_{bd} (d\nabla_c V_e) \text{ for any } V_b = \sigma_b^c V_c.
\]

The extrinsic curvature of \(\Sigma_t\) embedded in the ambient \(d\) dimensional spacetime (the constant \(r\) surfaces from the previous section) is

\[
\theta_{ab} := -\sigma_a^c \sigma_b^d (d\nabla_c \nu_d) = -d\nabla_a \nu_b - \nu_a \nu_b = -\frac{1}{2} \mathcal{L}_\nu \sigma_{ab}.
\]

This definition has a minus sign relative to the definition I normally use. This is for compatibility with the standard conventions in the literature.

Now we consider a ‘time flow’ vector field \(t^a\), which satisfies the condition

\[
t^a \partial_a t = 1.
\]

The vector \(t^a\) can be decomposed into parts normal and along \(\Sigma_t\) as

\[
t^a = \alpha u^a + \beta^a,
\]

where \(\alpha\) is the lapse function (2) and \(\beta^a := \sigma^a b^b\) is the shift vector. An important result in the derivations that follow relates the Lie derivative of a scalar or spatial tensor (one that is orthogonal to \(\nu^a\) in all of its indices) along the time flow vector field, to Lie derivatives along \(\nu^a\) and \(\beta^a\). Let \(S\) be a scalar. Then

\[
\mathcal{L}_t S = \mathcal{L}_\alpha u S + \mathcal{L}_\beta S = \alpha \mathcal{L}_u S + \mathcal{L}_\beta S.
\]

Rearranging this expression then gives

\[
\mathcal{L}_u S = \frac{1}{\alpha} \left( \mathcal{L}_t S - \mathcal{L}_\beta S \right).
\]
Similarly, for a spatial tensor with all lower indices we have

\[ \mathcal{L}_t W_{a...} = \alpha \mathcal{L}_u W_{a...} + \mathcal{L}_\beta W_{a...} . \]  

(11)

This is not the case when the tensor has any of its indices raised. In a moment, these identities will allow us to express certain Lie derivatives along \( u^a \) in terms of regular time derivatives and Lie derivatives along the shift vector \( \beta^a \).

Next, we construct the coordinate system that we will use for the decomposition of the equations of motion. The adapted coordinates \((t, x^i)\) are defined by

\[ \partial_t x^a := t^a . \]  

(12)

The \( x^i \) are \( d \) dimensional coordinates along the surface \( \Sigma_t \). If we define

\[ P^a_i := \frac{\partial x^a}{\partial x^i} , \]  

(13)

then it follows from the definition of the coordinates that \( P^a_i \partial_t t = 0 \) and we can use \( P^a_i \) to project tensors onto \( \Sigma_t \). For example, in the adapted coordinates the spatial metric, extrinsic curvature, and acceleration and shift vectors are

\[ \sigma_{ij} = P^a_i P^b_j \sigma_{ab} \]  

(14)

\[ \theta_{ij} = P^a_i P^b_j \theta_{ab} \]  

(15)

\[ a_j = P^b_j a^b \]  

(16)

\[ \beta_i = P^a_i \beta_a = P^a_i t^a . \]  

(17)

The line element in the adapted coordinates takes a familiar form:

\[ h_{ab} dx^a dx^b = h_{ab} \left( \frac{\partial x^a}{\partial t} dt + \frac{\partial x^a}{\partial x^i} dx^i \right) \left( \frac{\partial x^b}{\partial t} dt + \frac{\partial x^b}{\partial x^j} dx^j \right) \]  

(18)

\[ = h_{ab} \left( t^a dt + P^a_i dx^i \right) \left( t^b dt + P^b_j dx^j \right) \]  

(19)

\[ = t^a t^b dt^2 + 2 t^a dP^a_i dx^i + h_{ab} P^a_i P^b_j dx^i dx^j \]  

(20)

\[ = \left( - \alpha^2 + \beta^i \beta_i \right) dt^2 + 2 \beta^i dtdx^i + \sigma_{ij} dx^i dx^j \]  

(21)

\[ \Rightarrow h_{ab} dx^a dx^b = - \alpha^2 dt^2 + \sigma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt) . \]  

(22)

Thus, in the adapted coordinate system we can express the components of the (\( d \) dimensional) metric \( h_{ab} \) and its inverse \( h^{ab} \) as

\[ h_{ab} = \begin{pmatrix} -\alpha^2 + \beta^i \beta_i & \sigma_{ij} \beta^j \\ \sigma_{ij} \beta^j & \sigma_{ij} \end{pmatrix} \]  

(23)

\[ h^{ab} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{1}{\alpha^2} \beta^i \\ \frac{1}{\alpha^2} \beta^i & \sigma^{ij} - \frac{1}{\alpha^2} \beta^i \beta^j \end{pmatrix} \]  

(24)

\[ \det(h_{ab}) = -\alpha^2 \det(\sigma_{ij}) \]  

(25)

Obtaining the components of the inverse is a short algebraic calculation. Note that the spatial indices ‘\( i, j, \ldots \)’ in the adapted coordinates are lowered and raised using the spatial metric \( \sigma_{ij} \) and its inverse \( \sigma^{ij} \).
In adapted coordinates there are several results concerning the projections of Lie derivatives of scalars and
tensors which will be important in what follows. The first, which is trivial, is that the Lie derivative of a scalar
$S$ along the time-flow vector $t^a$ is just the regular time-derivative
\begin{equation}
\mathcal{L}_t S = t^a \partial_a S = \frac{\partial x^a}{\partial t} \frac{\partial S}{\partial x^a} = \partial_t S .
\end{equation}
(26)

Next, we consider the projector $P^a_i$ applied to the Lie derivative along $t^a$ of a general vector $W_a$, which gives
\begin{equation}
P^a_i \mathcal{L}_t W_a = \partial_t W_a \forall W_a .
\end{equation}
(27)

The important point is that this applies not just to spatial vectors but to any vector $W_a$, as a consequence of
the result
\begin{equation}
P^a_i \mathcal{L}_t u_a = 0 .
\end{equation}
(28)

Finally, we can show that the Lie derivative along $t^a$ of any contravariant spatial vector satisfies
\begin{equation}
P^i_a \mathcal{L}_t V^a = \partial_t V^i \forall V^i = P^i_a V^a .
\end{equation}
(29)

This follows from a lengthier calculation than what is required for the first two results.

Given these results, we can express various geometric quantities and their projections normal to and along
$\Sigma_t$ in terms of quantities intrinsic to $\Sigma_t$ and simple time derivatives. First, the extrinsic curvature is
\begin{equation}
\theta_{ij} = -\frac{1}{2} P^a_i P^b_j \mathcal{L}_u \sigma_{ab}
= -\frac{1}{2} P^a_i P^b_j \left( \frac{1}{\alpha} \left( \mathcal{L}_t \sigma_{ab} - \mathcal{L}_\beta \sigma_{ab} \right) \right)
\Rightarrow \theta_{ij} = -\frac{1}{2\alpha} \left( \partial_t \sigma_{ab} - \left( D_a \beta_b + D_b \beta_a \right) \right) .
\end{equation}
(30)\(31\)\(32\)

Since $\theta_{ab}$ is a spatial tensor, projections of its Lie derivative along $u^a$ can be expressed in a similar manner
\begin{equation}
P^a_i P^b_j \mathcal{L}_u \theta_{ab} = \frac{1}{\alpha} \left( \partial_t \theta_{ab} - \mathcal{L}_\beta \theta_{ab} \right) .
\end{equation}
(33)

Now we present the Gauss-Codazzi and related equations in adapted coordinates:
\begin{align}
P^a_i P^b_j (d R_{ab}) &= R_{ij} + \theta \theta_{ij} - 2 \theta^k_i \theta_{jk} - \frac{1}{\alpha} \left( \partial_t \theta_{ij} - \mathcal{L}_\beta \theta_{ij} \right) - \frac{1}{\alpha} D_i D_j \alpha
\tag{34}
P^a_i (d R_{ab} u^b) &= D_i \theta - D^i \theta_{ij} \tag{35}
d_{Rab} u^a u^b &= \frac{1}{\alpha} \left( \partial_t \theta - \beta^i \partial_i \theta \right) - \theta^i \partial_i \theta_{ij} + \frac{1}{\alpha} D_i D^i \alpha \tag{36}
d R &= R + \theta^2 + \theta^i \partial_i \theta_{ij} - \frac{2}{\alpha} \left( \partial_t \theta - \beta^i \partial_i \theta \right) - \frac{2}{\alpha} D_i D^i \alpha . \tag{37}
\end{align}

2 Converting to ADM Variables

The metric is often presented in the form
\begin{equation}
h_{ab} dx^a dx^b = h_{tt} dt^2 + 2 h_{ti} dt dx^i + h_{ij} dx^i dx^j .
\end{equation}
(38)
We would like to relate these components to the ADM variables: the lapse function $\alpha$, the shift vector $\beta_i$, and the spatial metric $\sigma_{ij}$. This is a fairly straightforward exercise in linear algebra. Comparing with (22), we first note that

$$\sigma_{ij} = h_{ij}.$$  \hfill (39)

The inverse spatial metric, $\sigma^{ij}$, is literally the inverse of $h_{ij}$, which is not the same thing as $h^{ij}$

$$\sigma^{ij} = (\sigma_{ij})^{-1} = (h_{ij})^{-1} \neq h^{ij}. \hfill (40)$$

For the shift vector we have

$$h_{ti} = \sigma_{ij} \beta^j \quad \rightarrow \quad \sigma^{ik} h_{tk} = \sigma^{ik} \sigma_{kl} \beta^l = \beta^i \quad \Rightarrow \quad \beta^i = \sigma^{ij} h_{tj}. \hfill (41)$$

Finally, for the lapse we obtain

$$\alpha^2 = \sigma^{ij} h_{tij} - h_{tt}. \hfill (43)$$