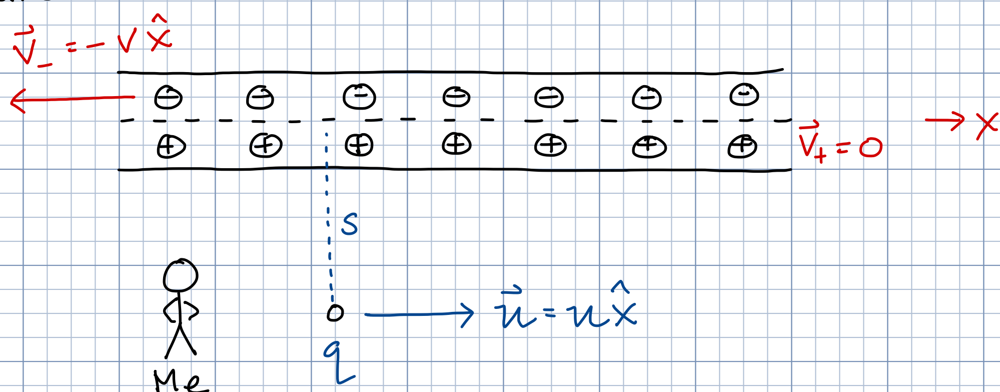
☒ Why Magnetic Fields?

- Suppose we knew about Electrostatics & Special Relativity. We'd be forced to conclude that a moving charge experiences a force that isn't explained by $\vec{F} = q\vec{E}$.
- (This assumes a few things about $\vec{E}$ due to moving charges, and strictly speaking E/S assumes the charges <u>aren't</u> moving. Don't worry: the conclusions here will be sound!)
- Consider a wire with both pos. & neg. charges in it. In my frame of reference the + charges are @ rest & the − charges are moving in the $-\hat{x}$ dir. @ speed v. <u>Also</u>, the spacing b/t the + charges & the spacing b/t the − charges is the same.



- Because of the spacing, the line charge density along the wire is <u>zero</u>. Any length of wire contains the same # of pos. & − charges:

$$\lambda_+ = \lambda \qquad \lambda_- = -\lambda \qquad \rightarrow \lambda_{TOT} = \lambda_+ + \lambda_- = 0$$

- Since $\lambda_{TOT} = 0$, the + & − charges in the wire don't create an electric field $\vec{E}$ outside the wire.

- Therefore, a charge q @ distance s from the wire feels no Coulomb force. (Assume that the + & − are spread throughout the wire so, unlike the figure, q isn't closer to a net positive charge.)

- But what if q was moving w/ velocity $\vec{u} = u\hat{x}$? In its frame of reference things would look different!

- From q's p.o.v. both the + & − charges are moving in the $-\hat{x}$ direction:

$$\vec{V}_+{}' = -u\hat{x} \qquad \vec{V}_-{}' = \frac{-u-v}{1+\frac{uv}{c^2}}\hat{x}$$

← Addition of velocities in SR!

- Because of length contraction, the spacing b/t + charges in q's frame of reference is:

$$l_+{}' = l\sqrt{1-\frac{V_+'^2}{c^2}} = l\sqrt{1-u^2/c^2}$$

← $l_+{}' < l$! They look closer together.

Since they appear closer together, q thinks the + charges have line charge density:

$$\lambda_+{}' = \frac{\lambda}{\sqrt{1-\frac{u^2}{c^2}}}$$

← $l_+{}' < l \Rightarrow$ Larger density than in my F.o.R.

- What about the − charges? They were moving already in my F.o.R., so their spacing in <u>their</u> rest frame is

$$l = \tilde{l}\sqrt{1-\frac{v^2}{c^2}} \Rightarrow \tilde{l} = \frac{l}{\sqrt{1-v^2/c^2}}$$

- So q sees the spacing b/t the − charges shortened to

$$\ell_-' = \tilde{\ell}\sqrt{1 - \frac{1}{c^2}\left(\frac{u+v}{1+\frac{uv}{c^2}}\right)^2} = \frac{\overbrace{\tilde{\ell}\sqrt{1-u^2/c^2}}^{\ell}\sqrt{1-v^2/c^2}}{1+\frac{uv}{c^2}}$$

$$= \ell\,\frac{\sqrt{1-u^2/c^2}}{1+\frac{uv}{c^2}}$$

In q's F.o.R., the spacing b/t the − charges means a line charge density of

$$\lambda_-' = -\lambda \times \frac{1+\frac{uv}{c^2}}{\sqrt{1-u^2/c^2}}$$

- Because of how we combine velocities in SR, the spacing b/t + charges & b/t − charges changes in different ways. As a result, the wire does not appear electrically neutral in q's F.o.R.!

$$\lambda_{TOT}' = \lambda_+' + \lambda_-' = \frac{\lambda}{\sqrt{1-u^2/c^2}} - \lambda\,\frac{(1+uv/c^2)}{\sqrt{1-u^2/c^2}}$$

$$\Rightarrow \lambda_{TOT}' = -\frac{\lambda}{\sqrt{1-u^2/c^2}}\,\frac{uv}{c^2}$$

− charges packed tighter than + charges, so λ is negative!

- Assuming Gauss's Law works the same way as it does in E/s, the charge q @ distance s feels an $\vec{E}$ field:

$$\vec{E}' = -\frac{uv}{c^2}\,\frac{1}{\sqrt{1-u^2/c^2}}\,\frac{\lambda}{2\pi\varepsilon_0}\,\frac{\hat{s}}{s}$$

$$\hookrightarrow \vec{F}' = q\vec{E}' = -\frac{uv}{c^2}\,\frac{1}{\sqrt{1-u^2/c^2}}\,\frac{q\lambda}{2\pi\varepsilon_0}\,\frac{\hat{s}}{s}$$

- This is a force that would pull a positive q in the negative $\hat{s}$ direction toward the wire.

- Before we go any further, what have we shown? If our understanding of E/S is correct then in my frame of reference the wire is electrically neutral ($\lambda_{tot} = 0$) and produces no electric field, and hence no force on q.

- But in q's frame of reference the wire carries a negative line charge density! This produces an electric field pointing towards the wire which would attract a positive q.

- So which one is it? There are many things that observers in different frames of reference disagree on, but a force perpendicular to $\vec{u}$ is not one of them!

- The answer, as you know, is that a current produces a magnetic field $\vec{B}$ which exerts a force on a moving charge q. If all you knew was Electrostatics and special relativity, you'd be forced to reach this conclusion in order to resolve the paradox.

- Indeed, you can see familiar structures in the expressions we found for $\vec{E}'$ & $\vec{F}'$ in q's frame of reference.

- In my frame of reference we'd say that the electrically neutral wire carries a current:

$$\vec{I} = (-\lambda)\,\vec{v}_- = (-\lambda)(-v\hat{x}) = \lambda v\,\hat{x}$$

- As we'll see, a long straight current produces a magnetic field w/ magnitude $|\vec{B}| \propto I/s$ @ distance s. A + charge moving parallel to the current w/ speed u should feel a force $|\vec{F}_B| \propto q u |\vec{B}|$ directed towards the wire.
- In that case, I'd expect the moving charge q to experience a force

$$\vec{F} \propto -\frac{q u \lambda v}{s} \hat{s}$$

In my frame of reference.
- When I switch over to q's frame of reference, special relativity tells me a force $\perp$ to $\vec{u}$ should transform as

$$\vec{F}' = \frac{1}{\sqrt{1 - u^2/c^2}} \vec{F}$$

This is exactly what we found!
- Now, in my F.o.R. we found $\vec{E} = 0$, and $\vec{F}$ was entirely due to a magnetic field $\vec{B}$ produced by the current $\vec{I} = \lambda v \hat{x}$.
- In the charges F.o.R. the current is

$$\vec{I}' = \lambda_+' \vec{v}_+' + \lambda_-' \vec{v}_-' = \frac{\lambda v \hat{x}}{\sqrt{1 - u^2/c^2}}$$

as expected, since the charge carriers appear closer together. This still produces a non-zero $\vec{B}'$, but no $\vec{F}_B'$ since q is @ rest in this F.o.R.

- Instead, the force $\vec{F}'$ comes from the electric field $\vec{E}'$ produced by $\lambda_{TOT}' \neq 0$.
- And this tells us something very important. Observers in different frames of reference disagree on what is charge density & what is current. And as a result, they naturally disagree on what is $\vec{E}$ & what is $\vec{B}$! But of course they agree on the physical consequences, such as a force on q.
- There's lots more we could do here, though to do it properly we'd need to introduce a bit more of the math behind special relativity.