WHY MAGNETIC FIELDS?

- Suppose we knew about Electrostatics & Special Relativity. We’d be forced to conclude that a moving charge experiences a force that isn’t explained by $F = qE$.

- (This assumes a few things about $E$ due to moving charges, and strictly speaking, E/S assumes the charges aren’t moving. Don’t worry; the conclusions here will be sound!)

- Consider a wire with both pos. & neg. charges in it. In my frame of reference, the + charges are @ rest & the - charges are moving in the - $\hat{x}$ dir. @ speed $v$. Also, the spacing b/t the + charges & the spacing b/t the - charges is the same.

  $\vec{v}_+ = -\vec{v}_- = \vec{v}$
  $\vec{v}_+ = \vec{v}_- = \vec{0}$

- Because of the spacing, the line charge density along the wire is zero. Any length of wire contains the same # of pos. & - charges:

  $\lambda_+ = \lambda_-$  $\rightarrow$  $\lambda_{tot} = \lambda_+ + \lambda_- = 0$
- Since \( \lambda_{\text{tor}} = 0 \), the + e- charges in the wire don't create an electric field \( \vec{E} \) outside the wire.

- Therefore, a charge \( q \) @ distance \( s \) from the wire feels no Coulomb force. (Assume that the + e- are spread throughout the wire so, unlike the figure, \( q \) isn't closer to a net positive charge.)

- But what if \( q \) was moving w/ velocity \( \vec{u} = u\hat{x} \)? In its frame of reference things would look different!

- From \( q \)'s p.o.v. both the + e- charges are moving in the - \( \hat{x} \) direction:

\[
\vec{v}_+ = -u\hat{x}, \quad \vec{v}_- = \frac{-u \mp \sqrt{u^2 - c^2}}{1 \mp \frac{u}{c^2}} \hat{x}
\]

- Because of length contraction, the spacing btw charges in \( q \)'s frame of reference is:

\[
l'_+ = l \sqrt{1 - \frac{u^2}{c^2}} = l \sqrt{1 - \frac{u^2}{c^2}} 
\]

Since they appear closer together, \( q \) thinks the + charges have line charge density:

\[
\lambda'_+ = \frac{\lambda}{\sqrt{1 - \frac{u^2}{c^2}}} \quad l'_+ < l \Rightarrow \text{Larger density than in my F.o.R.}
\]

- What about the - charges? They were moving already in my F.o.R., so their spacing in their rest frame is

\[
l = \tilde{\ell} \sqrt{1 - \frac{u^2}{c^2}} \Rightarrow \tilde{\ell} = \frac{l}{\sqrt{1 - \frac{u^2}{c^2}}}
\]
- So $q'$ sees the spacing bit the $-$ charges shortened to

$$l' = l \sqrt{1 - \frac{1}{C^2 (1 + \frac{\nu}{C})^2}} = l \sqrt{1 - \frac{\nu^2}{C^2}}$$

- In $q'$'s F.o.R., the spacing bit the $-$ charges means a line charge density of

$$\lambda' = -\lambda \times \frac{1 + \frac{\nu}{C}}{\sqrt{1 - \frac{\nu^2}{C^2}}}$$

- Because of how we combine velocities in SR, the spacing bit + charges and bit - charges changes in different ways. As a result, the wire does not appear electrically neutral in $q'$'s F.o.R.!

$$\lambda_{tot}' = \lambda_+ + \lambda_- = \lambda \times \frac{1 + \frac{\nu}{C}}{\sqrt{1 - \frac{\nu^2}{C^2}}} - \lambda \times \frac{1 + \frac{\nu}{C}}{\sqrt{1 - \frac{\nu^2}{C^2}}}$$

$$\Rightarrow \lambda_{tot}' = -\lambda \frac{\nu}{\sqrt{1 - \frac{\nu^2}{C^2}}}$$

- Assuming Gauss's Law works the same way as it does in E's, the charge $q$ at distance $s$ feels an $E$ field:

$$\vec{E}' = -\frac{\nu}{C^2 \sqrt{1 - \frac{\nu^2}{C^2}}} \frac{\lambda}{2 \pi \varepsilon_0} \frac{\hat{s}}{s}$$

$$\therefore \vec{F}' = q \vec{E}' = -\frac{\nu}{C^2 \sqrt{1 - \frac{\nu^2}{C^2}}} \frac{q \lambda}{2 \pi \varepsilon_0} \frac{\hat{s}}{s}$$
- This is a force that would pull a positive $q$ in the negative $\hat{s}$ direction toward the wire.

- Before we go any further, what have we shown? If our understanding of EIS is correct then in my frame of reference the wire is electrically neutral ($\lambda_{\text{tot}} = 0$) and produces no electric field, and hence no force on $q$.

- But in $q$’s frame of reference the wire carries a negative line charge density! This produces an electric field pointing towards the wire which would attract a positive $q$.

- So which one is it? There are many things that observers in different frames of reference disagree on, but a force perpendicular to $\vec{v}$ is not one of them!

- The answer, as you know, is that a current produces a magnetic field $\vec{B}$ which exerts a force on a moving charge $q$. If all you knew was Electrostatics and special relativity, you’d be forced to reach this conclusion in order to resolve the paradox.

- Indeed, you can see familiar structures in the expressions we found for $\vec{E}'$ & $\vec{E}$’ in $q$’s frame of reference.

- In my frame of reference we’d say that the electrically neutral wire carries a current:

$$\vec{I} = (-\vec{\lambda}) \cdot \hat{s} = (-\vec{\lambda}) (-\gamma \hat{\gamma} \vec{v} \times \hat{s}) = \gamma v \hat{\gamma} \hat{s}$$
- As we’ll see, a long straight current produces a magnetic field w/ magnitude \( |\mathbf{B}| \propto \frac{I}{S} \) @ distance \( s \). A + charge moving parallel to the current w/ speed \( v \) should feel a force \( |\mathbf{F}| \propto q |\mathbf{v} \times |\mathbf{B}| \) directed towards the wire.

- In that case, I’d expect the moving charge \( q \) to experience a force

\[
\mathbf{F} \propto -q \mathbf{v} \times \mathbf{v} \frac{S}{s}
\]

In my frame of reference.

- When I switch over to \( q \)’s frame of reference, special relativity tells me a force \( \mathbf{F} \) to \( \mathbf{v} \) should transform as

\[
\mathbf{F}' = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}} \mathbf{F}
\]

This is exactly what we found!

- Now, in my F.o.R. we found \( \mathbf{E} = 0 \), and \( \mathbf{F} \) was entirely due to a magnetic field \( \mathbf{B} \) produced by the current \( \mathbf{I} = \lambda \mathbf{v} \).

- In the charges F.o.R. the current is

\[
\mathbf{I}' = \lambda' \mathbf{v}' + \lambda' \mathbf{v}' = \frac{\lambda \mathbf{v}}{\sqrt{1 - u^2/c^2}}
\]

as expected, since the charge carriers appear closer together. This still produces a non-zero \( \mathbf{B}' \), but no \( \mathbf{F}' \) since \( q \) is @ rest in this F.o.R.
- Instead, the force $\mathbf{F}'$ comes from the electric field $\mathbf{E}'$ produced by $\mathbf{E}_\text{tor}' \neq 0$.

- And this tells us something very important. Observers in different frames of reference disagree on what is charge density $\rho$; what is current. And as a result, they naturally disagree on what is $\mathbf{E}$; what is $\mathbf{B}$! But of course they agree on the physical consequences, such as a force on $q$.

- There's lots more we could do here, though to do it properly we'd need to introduce a bit more of the math behind special relativity.