

## A Tricky Integral

One of the problems on Homework 4 contains an integral that is a little tricky. Problem number 3 requires you to integrate  $|\vec{r} - \vec{r}'|^{-1}$  over the volume of a cylinder, and this eventually leads to an integral of the form

$$\int dx \sqrt{x^2 + \alpha^2}. \quad (1)$$

How do we evaluate this integral? This is one of those times where you simply have to hammer at the integrand, using all the tricks you learned in your various calc classes to arrive at an answer. Let's run through the steps that are needed.

First, we use integration by parts to relate (1) to another integral that we will run into quite often. Applying 'u dv = d(uv) - v du' to the integrand gives

$$dx \sqrt{x^2 + \alpha^2} = d(x \sqrt{x^2 + \alpha^2}) - x d\sqrt{x^2 + \alpha^2} \quad (2)$$

$$= d(x \sqrt{x^2 + \alpha^2}) - x \frac{1}{2\sqrt{x^2 + \alpha^2}} 2x dx \quad (3)$$

$$= d(x \sqrt{x^2 + \alpha^2}) - \frac{x^2}{\sqrt{x^2 + \alpha^2}} dx \quad (4)$$

$$= d(x \sqrt{x^2 + \alpha^2}) - \frac{x^2 + \alpha^2 - \alpha^2}{\sqrt{x^2 + \alpha^2}} dx \quad (5)$$

$$= d(x \sqrt{x^2 + \alpha^2}) - \underbrace{\frac{\sqrt{x^2 + \alpha^2} dx}{\text{same as L.H.S.}}}_{\text{same as L.H.S.}} + \frac{\alpha^2}{\sqrt{x^2 + \alpha^2}} dx \quad (6)$$

$$dx \sqrt{x^2 + \alpha^2} = \frac{1}{2} d(x \sqrt{x^2 + \alpha^2}) + \frac{1}{2} \frac{\alpha^2}{\sqrt{x^2 + \alpha^2}} dx \quad (7)$$

$$\Rightarrow \int dx \sqrt{x^2 + \alpha^2} = \frac{1}{2} x \sqrt{x^2 + \alpha^2} + \frac{\alpha^2}{2} \int dx \frac{1}{\sqrt{x^2 + \alpha^2}}. \quad (8)$$

So now the question becomes: how do we evaluate the integral on the right-hand side of the last line? Here's where we start to deploy all the tricks you learn in calc. First, let's make a change of variables

$$x = \alpha \tan u \quad dx = \alpha \sec^2 u du \quad (9)$$

$$\rightarrow \int dx \frac{1}{\sqrt{x^2 + \alpha^2}} = \int du \frac{\alpha \sec^2 u}{\sqrt{\alpha^2 \tan^2 u + \alpha^2}} \quad (10)$$

$$= \int du \sec u. \quad (11)$$

Next, write  $\sec u$  as  $1/\cos u$  and then multiply the numerator and denominator by another factor of  $\cos u$ :

$$\int du \sec u = \int du \frac{1}{\cos u} = \int du \frac{\cos u}{\cos^2 u} = \int du \frac{\cos u}{1 - \sin^2 u}. \quad (12)$$

Another change of variables let's us write this in a form that does not involve trig functions

$$v = \sin u \quad dv = \cos u \, du \quad (13)$$

$$\int du \frac{\cos u}{1 - \sin^2 u} = \int dv \frac{1}{1 - v^2} \quad (14)$$

$$= \int dv \frac{1}{(1 - v)(1 + v)}. \quad (15)$$

In the last line I have factored the denominator:  $(1 - v^2) = (1 - v)(1 + v)$ . When the denominator can be written as the product of factors that are linear in the integration variables, you can try to use the technique of partial fractions to evaluate the integral. In this case we try to find numbers  $a$  and  $b$  such that

$$\frac{1}{(1 - v)(1 + v)} = \frac{a}{1 - v} + \frac{b}{1 + v}. \quad (16)$$

You can check for yourself that  $a = b = 1/2$ , so the integral (15) can be written as

$$\int dv \frac{1}{(1 - v)(1 + v)} = \frac{1}{2} \int dv \frac{1}{1 + v} + \frac{1}{2} \int dv \frac{1}{1 - v}. \quad (17)$$

Finally, we've obtained a few integrals that we know how to evaluate. Each of the integrals on the right-hand side gives us a log:

$$\int dv \frac{1}{(1 - v)(1 + v)} = \frac{1}{2} \ln(1 + v) - \frac{1}{2} \ln(1 - v) = \frac{1}{2} \ln \left( \frac{1 + v}{1 - v} \right). \quad (18)$$

Now we just need to work backwards, undoing each change of variables to get to an answer in terms of our original integration variable.

$$\frac{1}{2} \ln \left( \frac{1 + v}{1 - v} \right) = \frac{1}{2} \ln \left( \frac{1 + \sin u}{1 - \sin u} \right) \quad (19)$$

$$= \frac{1}{2} \ln \left( \frac{1 + \sin(\tan^{-1} \frac{x}{\alpha})}{1 - \sin(\tan^{-1} \frac{x}{\alpha})} \right) \quad (20)$$

$$= \frac{1}{2} \ln \left( \frac{1 + \frac{x}{\sqrt{x^2 + \alpha^2}}}{1 - \frac{x}{\sqrt{x^2 + \alpha^2}}} \right) \quad (21)$$

$$= \frac{1}{2} \ln \left( \frac{\sqrt{x^2 + \alpha^2} + x}{\sqrt{x^2 + \alpha^2} - x} \right) \quad (22)$$

$$\Rightarrow \int dx \frac{1}{\sqrt{x^2 + \alpha^2}} = \frac{1}{2} \ln \left( \frac{\sqrt{x^2 + \alpha^2} + x}{\sqrt{x^2 + \alpha^2} - x} \right). \quad (23)$$

This expression is correct, but for our purposes it isn't the most useful form of the integral. We can

simplify the answer just a bit more if we multiply by a convenient ‘factor of 1’ inside the log term

$$\frac{1}{2} \ln \left( \frac{\sqrt{x^2 + \alpha^2} + x}{\sqrt{x^2 + \alpha^2} - x} \right) = \frac{1}{2} \ln \left( \frac{\sqrt{x^2 + \alpha^2} + x}{\sqrt{x^2 + \alpha^2} - x} \cdot \frac{\sqrt{x^2 + \alpha^2} + x}{\sqrt{x^2 + \alpha^2} + x} \right) \quad (24)$$

$$= \frac{1}{2} \ln \left( \frac{(\sqrt{x^2 + \alpha^2} + x)^2}{x^2 + \alpha^2 - x^2} \right) \quad (25)$$

$$= \frac{1}{2} \ln \left( \frac{(\sqrt{x^2 + \alpha^2} + x)^2}{\alpha^2} \right) . \quad (26)$$

Since  $\log x^n = n \log x$  we get

$$\int dx \frac{1}{\sqrt{x^2 + \alpha^2}} = \ln \left( \frac{x + \sqrt{x^2 + \alpha^2}}{\alpha} \right) . \quad (27)$$

Remember, this isn’t the integral we were originally trying to evaluate! Going back to (8) and plugging in this result gives the answer we were looking for:

$$\int dx \sqrt{x^2 + \alpha^2} = \frac{1}{2} x \sqrt{x^2 + \alpha^2} + \frac{\alpha^2}{2} \ln \left( \frac{x + \sqrt{x^2 + \alpha^2}}{\alpha} \right)$$