

Separation of Variables and a Spherical Shell with Surface Charge

On Tuesday we worked out the electrostatic potential due to a spherical shell of radius R with a surface charge density $\sigma(\theta) = \sigma_0 \cos \theta$. This calculation involved several steps, so let's go back through it and make sure that everything we did is clear.

First, what are we trying to do? We want to find the potential due to the charge distribution described above. More specifically, we want to determine the potential inside the sphere, $r < R$, outside the sphere, $r > R$, and of course we would also like to know what happens at $r = R$. So the region \mathcal{V} where we want to determine the potential is “everywhere”, or all space. To pose a problem with a unique solution we need to describe any charge inside this region, and also specify the potential on the boundary of the region.¹ The charge inside the region has already been described: there is a surface charge density $\sigma(\theta) = \sigma_0 \cos \theta$ at $r = R$. What about the boundary of the region? In problems where we want to know the potential “everywhere”, the boundary is basically $r \rightarrow \infty$. Why is that? You can think of “everywhere” as the inside of a giant ball. The boundary of the ball is just the sphere that surrounds it. You make the ball bigger and bigger, so that it encloses more and more space, by letting its radius become arbitrarily large. This may sound a bit hand-wavy, but if we took the time we could make it more precise. The bottom line is that we need to say what the potential does as $r \rightarrow \infty$ if we want to find a unique expression for the potential everywhere.

Now, how do we approach this problem? As is usually the case, it's good to start by asking what sorts of simplifications we can make. Will the potential depend on all three coordinates as we move from point to point, or is it simpler than that? In this problem you get the feeling that it should be simpler. Why is that? This is an example of an *azimuthally symmetric* system. A system has azimuthal symmetry if all important aspects of the system – like the distribution of charge, or the potential on various surfaces – are symmetric about the same axis. That is the case here: all the charge is the same distance from the center of a sphere, and the amount (and sign) of the charge at a given point on the sphere only depends on the polar angle θ . So the charged spherical shell looks exactly the same from $\phi = 0$, $\phi = \pi/2$, $\phi = 11\pi/7$, or any other value of the azimuthal angle ϕ . How does azimuthal symmetry help us? As we saw in class, we have access to some powerful techniques for determining the potential in problems with azimuthal symmetry.²

At most of the points where we want to find the potential there is no charge, right? All the charge is on the shell at $r = R$, so when $r < R$ or $r > R$ the potential satisfies Laplace's equation

$$\nabla^2 V = 0 . \tag{1}$$

Our strategy in a problem like this to find the appropriate solutions of Laplace's equation in regions where there is no charge (like $r < R$ and $r > R$), and then “patch them together” where those regions overlap ($r = R$ in this case, where the charge is located). We're motivated to approach the problem this way because Laplace's equation for azimuthally symmetric problems can be solved quite generally

¹Remember, the “boundary” is a surface that encloses the region you are interested in. You may actually need more than one surface to enclose the region. For instance, to describe the potential between the walls of a cylindrical shell you would need to specify the potential on both the inner and outer walls of the cylinder.

²In fact, we also have access to powerful techniques that help with problems where there is much less symmetry. Separation of variables can be used to build up solutions to problems that depend on r , θ , and ϕ .

using separation of variables. In any region where there is no charge, the potential can be written as ³

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_{\ell} r^{\ell} + B_{\ell} \frac{1}{r^{\ell+1}} \right) P_{\ell}(\cos \theta) . \quad (2)$$

The A_{ℓ} and B_{ℓ} are constants, and the $P_{\ell}(\cos \theta)$ are Legendre polynomials.⁴ The coefficients A_{ℓ} and B_{ℓ} tell us how much of each separable solution $r^{\ell} P_{\ell}(\cos \theta)$ and $(1/r^{\ell+1}) P_{\ell}(\cos \theta)$ – to include to get the actual potential we want. Think of them like the components of a vector, which tell us the contributions needed in the \hat{x} , \hat{y} , and \hat{z} directions to get a specific vector. Now, we’ve seen in previous problems involving spheres, cylinders, etc that the potential may behave differently inside and outside the object. For example, outside a spherical shell with a *constant* surface charge density the potential falls off like $1/r$, but inside that sphere it is constant. So we expect that in a problem like this the potential might look different inside and outside the sphere. That means there are two different regions where we need to consider how the potential behaves. We’ll write the potential in the two regions as

$$V_{\text{in}}(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_{\ell}^{\text{in}} r^{\ell} + B_{\ell}^{\text{in}} \frac{1}{r^{\ell+1}} \right) P_{\ell}(\cos \theta) \quad (3)$$

$$V_{\text{out}}(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_{\ell}^{\text{out}} r^{\ell} + B_{\ell}^{\text{out}} \frac{1}{r^{\ell+1}} \right) P_{\ell}(\cos \theta) . \quad (4)$$

Notice that the potential has the same general *form* in both regions (powers of r time Legendre polynomials), but we assume that the details (the specific coefficients) are different. That is, the coefficients A_{ℓ}^{in} and A_{ℓ}^{out} probably aren’t the same, and likewise for B_{ℓ}^{in} and B_{ℓ}^{out} . The boundary conditions and other information for this problem will be used to identify these coefficients, leading us to the specific solution we want.

How do we figure out what the coefficients are? Well, what do we know about the problem? As we discussed in class, the potential should approach zero as we go very, very far away from the shell. After all, it’s just a blob of charge, so as we get far away we expect the potential to drop off (approach zero) at least as fast as the potential for a point charge. That tells us something about the potential outside the sphere; what about inside? Anything might be helpful, so what can we say? Well, at the very least we can say that since there is no charge inside the sphere, we don’t expect the potential to become too large anywhere. Does that make sense? Think about the potential due to a point charge: it becomes very large as you get close to the point charge, and in fact it is infinite right on top of the charge (because $1/z$ diverges at $z = 0$), but it doesn’t do that anywhere else. Since there are no charges inside the sphere, the potential should at least be finite (i.e., not divergent) everywhere inside the sphere. This sounds like I’m stating the obvious, but as we’ll see in a moment it turns out to be a useful observation. What about where “inside” and “outside” meet? We know the potential might behave differently inside and outside the sphere, but it *must* be continuous at the surface of the sphere where the two regions overlap. So our expressions for the potential inside and outside the sphere should agree at the surface of the sphere: $V_{\text{in}}(R, \theta) = V_{\text{out}}(R, \theta)$. Finally, there is one other thing we know that will be very important. The potential is always continuous, but not the electric

³This refers to a particular choice of origin and orientation of our coordinates! In this problem the origin is at the center of the sphere, and the polar angle was defined as soon as the charge density was described as $\sigma_0 \cos \theta$.

⁴If you don’t remember very much about Legendre polynomials, you should definitely read the relevant sections in Griffiths or your Math Methods book. Also, it’s *really* important to remember that the Legendre polynomials that appear in this general solution of Laplace’s equation are functions of $\cos \theta$, and not θ .

field. When a surface has a charge density on it, this will cause a discontinuity in the component of the electric field normal to the surface:

$$(E_{\text{out}}^{\perp} - E_{\text{in}}^{\perp}) \Big|_{\text{surface}} = \frac{\sigma}{\epsilon_0} \quad (5)$$

In this case the surface is a sphere, so the direction normal to the surface is \hat{r} . The r -component of the electric field is $-\partial V/\partial r$, which means that the potential has to satisfy⁵

$$-\frac{\partial V_{\text{out}}}{\partial r} \Big|_{r=R} - \left(-\frac{\partial V_{\text{in}}}{\partial r} \right) \Big|_{r=R} = \frac{\sigma_0 \cos \theta}{\epsilon_0}. \quad (6)$$

So, let's summarize what we know about the potential. Here are the four conditions we expect it to satisfy

- (1) $V_{\text{out}}(r, \theta) \rightarrow 0$ as $r \rightarrow \infty$
- (2) $V_{\text{in}}(r, \theta)$ is finite inside the sphere
- (3) $V_{\text{in}}(R, \theta) = V_{\text{out}}(R, \theta)$
- (4) $\left(\frac{\partial V_{\text{in}}}{\partial r} - \frac{\partial V_{\text{out}}}{\partial r} \right) \Big|_{r=R} = \frac{\sigma_0 \cos \theta}{\epsilon_0}$

Let's go through these conditions one at a time and see how they take us from the general solution (3) and (4) to the unique solution of this problem.

The first condition tells us that V_{out} should go to zero as $r \rightarrow \infty$. If we are very far away from the sphere it should just look like a little bit of charge. Since a point charge has a potential proportional to $1/r$, the potential for this charged spherical shell should approach zero *at least* as fast as $1/r$ if we move very far away from it. If we look at the general form (4) for the potential in the region $r > R$, we see that it contains terms proportional to both positive and negative powers of r , as well as a constant term (the $\ell = 0$ term). But if the non-negative powers of r were present then we wouldn't have a potential that goes to zero as $r \rightarrow \infty$! Therefore, the only way that (4) can be consistent with the requirement $V_{\text{out}} \rightarrow 0$ as $r \rightarrow \infty$ is if $A_{\ell}^{\text{out}} = 0$ for all values of ℓ . But be careful! Since this is a statement about the potential as $r \rightarrow \infty$, it is only telling us something about V_{out} . That is, we have learned that $A_{\ell}^{\text{out}} = 0$, but we have not learned anything about the coefficients A_{ℓ}^{in} and B_{ℓ}^{in} that appear in V_{in} .

Now let's move on to the second condition, which tells us that the potential is finite inside the sphere. The inside region corresponds to $0 \leq r < R$, so we need to examine the general solution of Laplace's equation to see if anything could go wrong in this region. There are two types of terms in (3): terms proportional to non-negative powers of r (the r^{ℓ} terms), and terms that involve negative powers of r (the $1/r^{\ell+1}$ terms). Do either of these do anything screwy in the region $0 \leq r < R$? Yes: any negative powers of r would diverge at $r = 0$. Since the potential is finite inside the sphere, it must be that the specific V_{in} we're after has no terms with negative powers of r . In other words, the coefficients of negative powers of r in (3) must be zero: $B_{\ell}^{\text{in}} = 0$.

At this point we've used the first two conditions to narrow down a general solution of Laplace's equation – we're on our way to finding the unique solution we want. Solutions consistent with these

⁵Remember that \vec{E} and V are related by $\vec{E} = -\vec{\nabla}V$. The minus sign is important in (6)!

first two conditions have the form

$$V_{\text{in}}(r, \theta) = \sum_{\ell=0}^{\infty} A_{\ell}^{\text{in}} r^{\ell} P_{\ell}(\cos \theta) \quad (7)$$

$$V_{\text{out}}(r, \theta) = \sum_{\ell=0}^{\infty} B_{\ell}^{\text{out}} \frac{1}{r^{\ell+1}} P_{\ell}(\cos \theta) . \quad (8)$$

To narrow things down further, let's look at the third condition. The potential must be continuous, so our expressions for the potential inside and outside the sphere must be the same at $r = R$, where inside and outside overlap. This means that (7) and (8) have to be equal at $r = R$

$$\sum_{\ell=0}^{\infty} A_{\ell}^{\text{in}} R^{\ell} P_{\ell}(\cos \theta) = \sum_{\ell=0}^{\infty} B_{\ell}^{\text{out}} \frac{1}{R^{\ell+1}} P_{\ell}(\cos \theta) . \quad (9)$$

This makes it seem like the coefficients A_{ℓ}^{in} and B_{ℓ}^{out} must be related somehow. To get the exact relation, we will use the fact that the Legendre polynomials $P_{\ell}(\cos \theta)$ are *orthogonal* on the interval $0 \leq \theta \leq \pi$. Since we're trying to determine the potential everywhere ($0 \leq r < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$), the equality (9) has to be true over these same values of θ . If we move both series to one side of the equation we get

$$\sum_{\ell=0}^{\infty} \left(A_{\ell}^{\text{in}} R^{\ell} - B_{\ell}^{\text{out}} \frac{1}{R^{\ell+1}} \right) P_{\ell}(\cos \theta) = 0 . \quad (10)$$

Orthogonality of the Legendre polynomials means that this can be true for all $0 \leq \theta \leq \pi$ if and only if every term in the series is zero.⁶ Since each term in (10) has to be zero on its own, we see that

$$A_{\ell}^{\text{in}} R^{\ell} - B_{\ell}^{\text{out}} \frac{1}{R^{\ell+1}} = 0 \quad (11)$$

$$\Rightarrow B_{\ell}^{\text{out}} = R^{2\ell+1} A_{\ell}^{\text{in}} . \quad (12)$$

This is true for all values of ℓ , so for all the terms in the series (8).

Important: Starting from (9), use the orthogonality relation for Legendre polynomials to prove that (11) is true. First multiply both sides of (9) by $P_k(\cos \theta) \sin \theta$ (where k is arbitrary) and then use

$$\int_0^{\pi} d\theta \sin \theta P_{\ell}(\cos \theta) P_k(\cos \theta) = \frac{2}{2\ell + 1} \delta_{\ell,k} . \quad (13)$$

So now, after applying the first three conditions on the potential, we have

$$V_{\text{in}}(r, \theta) = \sum_{\ell=0}^{\infty} A_{\ell}^{\text{in}} r^{\ell} P_{\ell}(\cos \theta) \quad (14)$$

$$V_{\text{out}}(r, \theta) = \sum_{\ell=0}^{\infty} A_{\ell}^{\text{in}} \frac{R^{2\ell+1}}{r^{\ell+1}} P_{\ell}(\cos \theta) . \quad (15)$$

⁶It's exactly the same as saying that $A_x \hat{x} + A_y \hat{y} + A_z \hat{z} = 0$ can only be true if the components A_x , A_y , and A_z are all zero. Since \hat{x} , \hat{y} , and \hat{z} are orthogonal, you can't have non-zero x , y , and z components somehow adding up to zero.

We're almost done! We started with two infinite series, with each series having two unspecified sets of coefficients in it. After applying our first three conditions we've narrowed that down to one set of unknown coefficients. We've gone from $4 \times \infty$ unknowns to $1 \times \infty$ unknowns!

To determine the remaining unknowns in (14) and (15) we will use the fourth and final condition on the potential. We know that the electric field should be discontinuous at $r = R$ since there is a surface charge there. Specifically, we have

$$\left(\frac{\partial V_{\text{in}}}{\partial r} - \frac{\partial V_{\text{out}}}{\partial r} \right) \Big|_{r=R} = \frac{\sigma_0 \cos \theta}{\epsilon_0} . \quad (16)$$

This shouldn't be too hard to implement! First, let's take the r -derivatives of (14) and (15).

$$\frac{\partial V_{\text{in}}(r, \theta)}{\partial r} = \sum_{\ell=0}^{\infty} \ell A_{\ell}^{\text{in}} r^{\ell-1} P_{\ell}(\cos \theta) \quad (17)$$

$$\frac{\partial V_{\text{out}}(r, \theta)}{\partial r} = \sum_{\ell=0}^{\infty} -(\ell+1) A_{\ell}^{\text{in}} \frac{R^{2\ell+1}}{r^{\ell+2}} P_{\ell}(\cos \theta) . \quad (18)$$

Evaluating these expressions at $r = R$ and putting them in (16), we get

$$\sum_{\ell=0}^{\infty} (2\ell+1) A_{\ell}^{\text{in}} R^{\ell-1} P_{\ell}(\cos \theta) = \frac{\sigma_0 \cos \theta}{\epsilon_0} . \quad (19)$$

Now all that's left is to use the orthogonality relations for the Legendre polynomials to figure out what the unknown coefficients A_{ℓ}^{in} are. First we multiply both sides of (19) by a factor of $\sin \theta$ and an arbitrary Legendre polynomial $P_k(\cos \theta)$

$$\sum_{\ell=0}^{\infty} (2\ell+1) A_{\ell}^{\text{in}} R^{\ell-1} P_{\ell}(\cos \theta) P_k(\cos \theta) \sin \theta = \frac{\sigma_0}{\epsilon_0} P_k(\cos \theta) \cos \theta \sin \theta . \quad (20)$$

Next, integrate both sides of this equation with respect to θ , from 0 to π

$$\sum_{\ell=0}^{\infty} (2\ell+1) A_{\ell}^{\text{in}} R^{\ell-1} \int_0^{\pi} d\theta P_{\ell}(\cos \theta) P_k(\cos \theta) \sin \theta = \frac{\sigma_0}{\epsilon_0} \int_0^{\pi} d\theta P_k(\cos \theta) \cos \theta \sin \theta . \quad (21)$$

The integral on the left-hand side of this equation is zero *unless* $\ell = k$, in which case it is $2/(2k+1)$

$$\int_0^{\pi} d\theta P_{\ell}(\cos \theta) P_k(\cos \theta) \sin \theta = \frac{2}{2k+1} \delta_{\ell,k} . \quad (22)$$

So for the left-hand side of (21) we get

$$\sum_{\ell=0}^{\infty} (2\ell+1) A_{\ell}^{\text{in}} R^{\ell-1} \int_0^{\pi} d\theta P_{\ell}(\cos \theta) P_k(\cos \theta) \sin \theta = \sum_{\ell=0}^{\infty} (2\ell+1) A_{\ell}^{\text{in}} R^{\ell-1} \frac{2}{2k+1} \delta_{\ell,k} \quad (23)$$

$$= 2 A_k^{\text{in}} R^{k-1} . \quad (24)$$

What about the right-hand side of (21)? We've seen this integral before, and we evaluated it by noting that $\cos \theta$ is the same thing as $P_1(\cos \theta)$. Then

$$\frac{\sigma_0}{\epsilon_0} \int_0^{\pi} d\theta P_k(\cos \theta) \cos \theta \sin \theta = \frac{\sigma_0}{\epsilon_0} \int_0^{\pi} d\theta P_k(\cos \theta) P_1(\cos \theta) \sin \theta \quad (25)$$

$$= \frac{\sigma_0}{\epsilon_0} \frac{2}{2k+1} \delta_{k,1} . \quad (26)$$

So comparing our results for the two sides of (21), we get

$$2 A_k^{\text{in}} R^{k-1} = \frac{\sigma_0}{\varepsilon_0} \frac{2}{2k+1} \delta_{k,1} . \quad (27)$$

If $k = 0$, or $k = 2$, or k is equal to any integer other than “1”, then the right-hand side is zero. This means that A_k^{in} is zero for all those values of k ! But if $k = 1$ then the right-hand side is non-zero, so we get a non-zero value for A_1^{in} . We find

$$A_1^{\text{in}} = \frac{\sigma_0}{3\varepsilon_0} \quad (28)$$

$$A_k^{\text{in}} = 0 \quad \forall k \neq 1 . \quad (29)$$

That’s it! We used our fourth condition on the potential to determine the remaining unknowns in our expressions (14) and (15) for the potential. When we re-write those expressions using our results for A_k^{in} , we get

$$V_{\text{in}}(r, \theta) = \frac{\sigma_0}{3\varepsilon_0} r P_1(\cos \theta) \quad (30)$$

$$V_{\text{out}}(r, \theta) = \frac{\sigma_0}{3\varepsilon_0} \frac{R^3}{r^2} P_1(\cos \theta) \quad (31)$$

Putting everything together, our final answer is

$$V(r, \theta) = \begin{cases} \frac{\sigma_0}{3\varepsilon_0} r \cos \theta & r \leq R \\ \frac{\sigma_0}{3\varepsilon_0} \frac{R^3}{r^2} \cos \theta & r \geq R \end{cases} \quad (32)$$

There are a few final points to make. First, notice that the expressions for the potential inside and outside the sphere agree at $r = R$. We made a point of enforcing that with our third condition, but it’s always good to check – we could have made an algebra mistake somewhere along the way. Now, in both those regions we were solving Laplace’s equation. But really we’ve done more than that, because there is charge at $r = R$. So by joining our two solutions of Laplace’s equation together in the right way at the right place, we’ve actually found the correct solution of *Poisson’s equation* for a spherical shell with given surface charge density. Second, we didn’t actually use the specific form of the surface charge density until we got to (19). If we had a spherical shell with a different (azimuthally symmetric) surface charge density $\sigma(\theta)$ we could just jump right to the expressions (14) and (15) for the potential, and determine the unknown coefficients as above, starting from the condition

$$\sum_{\ell=0}^{\infty} (2\ell+1) A_\ell^{\text{in}} R^{\ell-1} P_\ell(\cos \theta) = \frac{\sigma(\theta)}{\varepsilon_0} . \quad (33)$$

Third, let’s consider what V looks like far away from the sphere. We see from (32) that the potential is proportional to $1/r^2$ outside the sphere. Can that be right? This is just a blob of charge, so if we’re far away it looks like a point. Shouldn’t we get a $1/r$ potential? Not so fast! Look at the total charge on the sphere:

$$q_{\text{tot}} = \int da \sigma = \int_0^{2\pi} d\phi \int_0^\pi d\theta R^2 \sin \theta \sigma_0 \cos \theta = 2\pi R^2 \int_0^\pi d\theta \sin \theta \cos \theta = 0 . \quad (34)$$

So to a first approximation the potential outside the sphere does look just like a point charge, but it's a point charge with zero charge! This just tells us that there is no $1/r$ term in the potential. Finally and most importantly, we found a *unique* solution to our problem, as promised. How do we know? Because we started with 4 sets of unknowns (A_ℓ^{in} , B_ℓ^{in} , A_ℓ^{out} , and B_ℓ^{out}) and we finished with zero unknowns. Everything has been nailed down, and there are no other possibilities. If we got to the end of the problem and still had unknowns left, it would mean that we forgot some important condition on the potential.