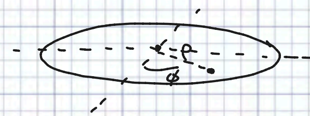
INTRO

- At the end of our last section we worked out the Laplacian in any OCS.

- The most basic, idealized eqn. for a circular vibrating drum or membrane looks just like the wave eqn for a string, w/ ∇^2 replacing the d^2/dx^2.

$$\underbrace{\nabla^2 z(\rho,\phi,t)}_{} - \frac{1}{v^2}\frac{d^2 z(\rho,\phi,t)}{dt^2} = 0$$



Height of pt. (ρ,ϕ) above/below Eq. @ time t

- There are lots of solutions to this PDE, and we'll talk about this in more detail later. But for now, let's focus on 'radial solutions' — sol'ns that don't depend on ϕ. Then:

$$\frac{d^2 z(\rho,t)}{d\rho^2} + \frac{1}{\rho}\frac{dz(\rho,t)}{d\rho} - \frac{1}{v^2}\frac{d^2\rho}{dt^2} = 0$$

- Next month we'll discuss techniques for solving this sort of PDE. For now, we'll simplify again & only consider what we'll call 'NORMAL MODES.' They take the form

$$z(\rho,t) = R(\rho)\,T(t)$$

Most sol'ns DO NOT look like this! More later.

- A sol'n of this form only works if $R(\rho)$ & $T(t)$ satisfy:

$$\frac{d^2T}{dt^2} = -k^2v^2T \qquad \rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + k^2\rho^2R = 0$$

For some CONSTANT k!

- The 1^{st} eqn is pretty easy, right? T could be a cos or sin, & since the eqn. is <u>linear</u> its general sol'n is

$$T(t) = a_1 \cos(kvt) + a_2 \sin(kvt)$$

kv is ω!

- But what about the 2^{nd} eqn? Doesn't look familiar, and if we try something simple like $R(\rho) \sim \rho^{\text{const.}}$ that doesn't work.

- But it has to be <u>something</u>. So whatever it is, let's assume it can be written as a series around $\rho = 0$ (so a Maclaurin series). That seems reasonable, right? It's just a vibrating drum head. That seems like it should be well-behaved & therefore admit some kind of series description.

ASSUME:
$$R(\rho) = c_0 + c_1\rho + c_2\rho^2 + c_3\rho^3 + c_4\rho^4 + c_5\rho^5 + \dots$$
$$R'(\rho) = 0 + c_1 + 2c_2\rho + 3c_3\rho^2 + 4c_4\rho^3 + 5c_5\rho^4 + \dots$$
$$R''(\rho) = 0 + 0 + 2c_2 + 6c_3\rho + 12c_4\rho^2 + 20c_5\rho^3 + \dots$$

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + k^2\rho^2 R = 0$$

$$\hookrightarrow 2c_2\rho^2 + 6c_3\rho^3 + 12c_4\rho^4 + 20c_5\rho^5 + \dots$$
$$+ c_1\rho + 2c_2\rho^2 + 3c_3\rho^3 + 4c_4\rho^4 + 5c_5\rho^5 + \dots$$
$$+ k^2c_0\rho^2 + k^2c_1\rho^3 + k^2c_2\rho^5 + \dots = 0$$

$$\Rightarrow 0 = c_1\cdot\rho + (4c_2 + k^2c_0)\rho^2 + (9c_3 + k^2c_1)\rho^3$$
$$+ (16c_4 + k^2c_2)\rho^4 + (25c_5 + k^2c_3)\rho^5 + \dots$$

- Okay, for this to work – for $R(\rho)$ to have the sort of power series description we assumed – this series we get for the ODE must vanish term-by-term.

- That is, the Maclaurin series for <u>zero</u> is

$$0 = 0 + 0 \cdot \rho + 0 \cdot \rho^2 + 0 \cdot \rho^3 + 0 \cdot \rho^4 + \ldots$$

So we have:

$$c_1 = 0$$

$$4c_2 + k^2 c_0 = 0 \Rightarrow c_2 = -\tfrac{1}{4} k^2 c_0$$

$$9c_3 + k^2 c_1 = 0 \Rightarrow c_3 = 0 \text{ b/c } c_1 = 0$$

$$16 c_4 + k^2 c_2 = 0 \Rightarrow c_4 = -\tfrac{1}{16} k^2 c_2 = \tfrac{1}{64} k^2 c_0$$

$$25 c_5 + k^2 c_3 = 0 \Rightarrow c_5 = 0 \text{ b/c } c_3 = 0$$

$$36 c_6 + k^2 c_4 = 0 \Rightarrow c_6 = -\tfrac{1}{36} k^2 c_4 = -\tfrac{1}{2304} k^2 c_0$$

- All the odd powers have coefficient zero. Nothing in the eqn. tells us what c_0 is, but subsequent terms are:

$$R(\rho) = c_0 \times \left(1 - \tfrac{1}{4} k^2 \rho^2 + \tfrac{1}{64} k^4 \rho^4 - \tfrac{1}{2304} k^6 \rho^6 + \ldots\right)$$

- We have one unknown here: c_0. And that makes sense – when we solve the wave eqn for a string it doesn't give us the amplitude!

- (Wait – shouldn't we expect <u>two</u> unknowns for a 2nd order eqn? Yes, but we eliminated one assuming Maclaurin!)

- In this section we're going to learn how to solve ODEs (& some other eqns) by assuming that the sol'n can be written as a series. Could be a Maclaurin series, but we'll also see Taylor series & other sorts of series as well.

- Let's go back to example we just did:

$$0 = \rho^2 \times (4c_2 + k^2 c_0) + \rho^4 \times (16 c_4 + k^2 c_2) + \rho^6 \times (36 c_6 + k^2 c_4) + \rho^8 \times (64 c_8 + k^2 c_6) + \dots$$

- Clear that coeff. of ρ^{2n+2} always related to coeff. of ρ^{2n} by:

$$(2n+2)^2 c_{2n+2} + k^2 c_{2n} = 0$$

"RECURRENCE RELATION"

- A RECURRENCE RELATION tells us how terms in our series relate to the previous terms. Once we have the recurrence rel'n we solve it iteratively to get an EXPLICIT form of the c_n.

$$c_{2n+2} = -\frac{k^2}{(2n+2)^2} c_{2n}$$

$$c_2 = -\frac{k^2}{2^2} c_0$$

$$c_4 = -\frac{k^2}{4^2} c_2 = (-1)^2 \frac{k^4}{4^2 2^2} c_0$$

$$c_6 = -\frac{k^2}{6^2} c_4 = (-1)^3 \frac{k^6}{6^2 4^2 2^2} c_0$$

$$\vdots$$

$$c_{2n} = (-1)^n \frac{k^{2n}}{2^{2n}(n!)^2}$$

Sum gives a "BESSEL FUNCTION"

$$J_0(k\rho) = \sum_{n=0}^{\infty} (-1)^n \frac{(k\rho)^{2n}}{2^{2n}(n!)^2}$$

- Before we can learn to identify & solve recurrence relations, we'll need to review some techniques for working with sums.
- After that, we'll apply this technique to general equations. Then we'll consider some equations that frequently arise in physics, and the families of functions – defined as finite or infinite series – that solve them.

Working With Sums

- Sometimes a series sol'n of a DE has a finite number of terms, and sometimes it's an infinite sum.
- Either way, we'll always start w/ an ∞ series, work out recurrence relations based on the DE, and see what it gives us.
- So it's useful to review some techniques for working w/ ∞ series, especially ways of rewriting them or combining two or more series into a single series.
- Most of this is covered in section 12.2 of F & F.
- Sometimes we see a series where the terms alternate sign. We account for this w/ a factor like $(-1)^n$ inside the sum:

1, −1, 1, −1, 1, −1, ...

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n}$$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

$$\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{3}\right)^n$$

Could also use $(-1)^{n-1}$!

- It's also common to encounter a sum that involves only even or odd powers of some quantity. For instance, maybe only even powers of x appear. We'll label each term's coefficient in the sum w/ its power, so:

$$c_0 + c_2 x^2 + c_4 x^4 + c_6 x^6 + \dots$$

- How do we write this as an ∞ sum? Since any even # can be written as $2n$ for some $n = 0, 1, 2, \dots$

$$c_0 + c_2 x^2 + c_4 x^4 + c_6 x^6 + \dots = \sum_{n=0}^{\infty} c_{2n} x^{2n}$$

- Likewise, for a sum w/ only odd powers of x, we can write any odd # as $2n+1$ w/ $n = 0, 1, 2, \dots$. So:

$$c_1 + c_3 x^3 + c_5 x^5 + c_7 x^7 + \dots = \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1}$$

- And sometimes it is useful to break a sum w/ both even & odd terms into separate even & odd sums:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

$$= \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1}$$

- But the sort of manipulation we'll do most often is 're-indexing' one or more sums so they can be combined into a single sum.

- For instance, we might find ourselves adding together two ∞ series of the form

$$\sum_{n=0}^{\infty} n \cdot (n-1) \cdot c_n x^{n-2} + \sum_{n=0}^{\infty} \alpha\, c_n \cdot x^n + (\text{OTHER STUFF})$$

Same coefficients show up in both but w/ diff. powers of x

- Let's write out the 1st several terms of both series.

1st : $0 \cdot c_0 \cdot x^{-2} + 0 \cdot c_1 \cdot x^{-1} + 2 \cdot 1 \cdot c_2 x^0 + 3 \cdot 2 \cdot c_3 x^1 + 4 \cdot 3 \cdot c_4 x^2 + \ldots$

2nd : $\alpha c_0 \cdot x^0 + \alpha c_1 x^1 + \alpha c_2 x^2 + \alpha c_3 x^3 + \alpha c_4 x^4 + \ldots$

- The same coefficients appear in both series, but w/ different powers of x. Also, even though the 1st series looks like it starts @ x^{-2}, the 1st two terms are zero. So we should be able to write all this as a single sum, right?

1st + 2nd : $(2c_2 + \alpha c_0) x^0 + (6 c_3 + \alpha c_1) x^1 + (12 c_4 + \alpha c_2) \cdot x^2$

$+ (20 c_5 + \alpha c_3) x^3 + \ldots$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1) c_{n+2} + \alpha c_n \right) x^n$$

- How did I get the factor of $(n+2)(n+1)$? Did I guess it? No, I RE-INDEXED the 1st sum. First:

$$\sum_{n=0}^{\infty} n \cdot (n-1) \cdot c_n x^{n-2} = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot c_n \cdot x^{n-2}$$ ← B/c $n=0$ & $n=1$ terms are zero.

- Now, I want this to be a sum that starts @ $n=0$ & has powers of x^n, so I can combine it w/ the other sum. I do this by shifting $n \to n+2$.

$$\sum_{n=2}^{\infty} \underbrace{n \cdot (n-1) \cdot c_n \cdot x^{n-2}}_{\text{Every } n \text{ here...}} = \sum_{n=0}^{\infty} \underbrace{(n+2)(n+1) c_{n+2} x^n}_{\text{... is replaced by } n+2 \text{ here.}}$$

$$\Rightarrow \sum_{n=0}^{\infty} n \cdot (n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

(Write out 1st several terms & check!)

- Combining sums like this will be one of our main tools for finding recurrence relations b/t terms in a series sol'n of some eqn.

- For instance, suppose I told you that when I add those series together they should cancel. Then:

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)\, c_{n+2} + \alpha c_n \right) x^n = 0$$

- If a power series is supposed to be zero $\forall\, x$ then it must vanish term-by-term. For instance, the x^3 power can't cancel the x^4 power. So:

$$(n+2)(n+1)\, c_{n+2} + \alpha c_n = 0 \Rightarrow c_{n+2} = -\frac{\alpha}{(n+2)(n+1)} c_n$$

- Neither c_0 nor c_1 are determined by this condition. But all the subsequent terms are.

$$c_2 = -\frac{\alpha}{2} c_0 \qquad c_4 = -\frac{\alpha}{12} c_2 = \frac{\alpha}{24} c_0 \qquad c_6 = -\frac{\alpha}{720} c_0 \quad \text{etc.}$$

$$c_3 = -\frac{\alpha}{6} c_1 \qquad c_5 = -\frac{\alpha}{20} c_3 = \frac{\alpha}{120} c_1 \qquad c_7 = -\frac{\alpha}{5040} c_1$$

$$\hookrightarrow \sum_{n=0}^{\infty} c_n x^n = c_0 \times \left(1 - \frac{\alpha}{2} x^2 + \frac{\alpha}{24} x^4 - \frac{\alpha}{720} x^6 + \dots \right)$$
$$+ c_1 \times \left(1 - \frac{\alpha}{6} x^3 + \frac{\alpha}{120} x^5 - \frac{\alpha}{5040} x^7 + \dots \right)$$

BTW, this is $c_0 \cos(\alpha x) + c_1 \sin(\alpha x)$!

- Sometimes we add two or more sums so they have the same powers of x, but they don't "overlap" – one series has one or more extra terms.

$$\underbrace{\sum_{n=0}^{\infty} c_n \cdot x^{n+1}}_{c_0 x^1 + c_1 x^2 + c_2 x^3 + c_3 x^4 + \dots} + \underbrace{\sum_{n=0}^{\infty} n \cdot c_n \cdot x^{n-1}}_{0 \cdot c_0 \cdot x^{-1} + 1 \cdot c_1 \cdot x^0 + 2 c_2 x^1 + 3 c_3 x^2 + 4 c_4 x^3 + \dots} = ?$$

RE-INDEX 2ND SUM: $n \to n+2$

$$\sum_{n=1}^{\infty} n\, c_n x^{n-1} \to \sum_{n=-1}^{\infty} (n+2) c_{n+2} x^{n+1}$$

$$= c_1 x^0 + \sum_{n=0}^{\infty} (n+2) c_{n+2} x^{n+1}$$

Series Solutions of Linear ODEs

- As described in our motivating example, we're going to solve certain differential equations by assuming the sol'n can be written as some sort of power series.

- For now, we'll focus on Maclaurin series — an expansion around $x=0$ or $\rho=0$, etc — but later we'll consider other sorts of power series.

- The eqns we'll study have the form

$$y''(x) + f(x)\,y'(x) + g(x)\,y(x) = h(x)$$

where $f(x), g(x), \&\, h(x)$ are <u>ANALYTIC</u> functions. (They are differentiable @ $x=0$ & have Maclaurin series of their own. So x^2 or e^x is okay, but $\sqrt{x}$ isn't.)

- We could also allow more (or fewer) derivatives, & assume all coefficients are analytic, and this approach would still work! But we'll mostly look @ 2nd order linear ODEs.

- The strategy is easy: Assume $y(x)$ can be written as a series, plug it in to the eqn, & find recurrence rel'ns for the terms in the series.

<u>Ex</u> $y'(x) + 2\alpha x\, y(x) = 0$ w/ $\alpha = \text{const.}$

$$\left.\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^n \\ y'(x) &= \sum_{n=0}^{\infty} n\, c_n x^{n-1} \end{aligned}\right\} \sum_{n=0}^{\infty} n\, c_n x^{n-1} + \sum_{n=0}^{\infty} 2\alpha c_n x^{n+1} = 0$$

$$\Rightarrow 0 = 0\cdot c_0\cdot x^{-1} + 1\cdot c_1\cdot x^0 + 2\cdot c_2\cdot x^1 + 3\cdot c_3\cdot x^2 + 4\cdot c_4\cdot x^3 + \dots$$
$$+ 2\alpha\cdot\left(c_0 x^1 + c_1 x^2 + c_2 x^3 + c_3 x^4 + \dots\right)$$
$$= c_1\cdot x^0 + (2c_2 + 2\alpha c_0)x^1 + (3c_3 + 2\alpha c_1)x^2$$
$$+ (4c_4 + 2\alpha c_2)x^3 + (5c_5 + 2\alpha c_3)x^4 + \dots$$

- So the eqn. is satisfied if

$$c_1 = 0 \qquad c_2 = -\alpha c_0 \qquad c_3 = -\tfrac{2}{3}\alpha c_1 = 0 \qquad c_4 = -\tfrac{\alpha}{2}c_2 = \tfrac{\alpha^2}{2}c_0$$
$$c_5 = -\tfrac{2}{5}\alpha c_3 = 0 \qquad c_6 = -\tfrac{\alpha}{3}c_4 = -\tfrac{\alpha^3}{3!}c_0 \qquad \dots$$

$$\hookrightarrow y(x) = c_0 \times \left(1 - \alpha x^2 + \frac{\alpha^2}{2}x^4 - \frac{\alpha^3}{3!}x^6 + \dots\right)$$

- There's a pattern here!

1st order ODE ⇒ One unknown!

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{1}{n!}(-\alpha x^2)^n = c_0\, e^{-\alpha x^2}$$

- Of course, we <u>knew</u> this was the sol'n:

$$y' + 2\alpha x y = 0 \Rightarrow \frac{y'}{y} = -2\alpha x \Rightarrow \frac{d}{dx}\ln y = -2\alpha x$$
$$\Rightarrow \ln y = -\alpha x^2 + \text{const} \Rightarrow y(x) = c_0\, e^{-\alpha x^2}$$

- But we don't want to rely on spotting patterns in the c_n, so can we <u>derive</u> c_n by re-indexing the two ∞ series & working out a recurrence relation?

$$\underbrace{\sum_{n=0}^{\infty} n\, c_n\, x^{n-1}}_{\text{The } n=0 \text{ term vanishes}} + 2\alpha \underbrace{\sum_{n=0}^{\infty} c_n\, x^{n+1}}_{\text{Can we shift summation variable so the power of } x \text{ is } n-1\text{?}} = 0$$

$$\sum_{n=1}^{\infty} n\, c_n\, x^{n-1} + 2\alpha \overbrace{\sum_{n-2=0}^{\infty} c_{n-2}\, x^{n-1}}^{n \to n-2} = 0$$

$$\to \sum_{n=1}^{\infty} n\, c_n\, x^{n-1} + 2\alpha \sum_{n=2}^{\infty} c_{n-2}\, x^{n-1} = 0$$

Same form, but one starts @ $n=1$ & other starts @ $n=2$.

$$\to 1 \cdot c_1 \cdot x^0 + \sum_{n=2}^{\infty} \left(n\, c_n + 2\alpha\, c_{n-2} \right) x^{n-1} = 0$$

- Once we have it in this form we learn the following:

  (1) c_0 only shows up when $n=2$ $(2c_2 + 2\alpha c_0 = 0)$ so it is not determined by the eqn.

  (2) $c_1 = 0$, b/c that's the only way the constant term on the LHS of the eqn can vanish.

  (3) The recurrence rel'n for the remaining coefficients is

$$2\alpha\, c_{n-2} + n\, c_n = 0 \implies c_n = -\frac{2\alpha}{n} c_{n-2}$$

- Since $c_1 = 0$, it follows that all the odd c_{2n+1} are also zero. What about the even coefficients? That's when $n = 2k$ for $k = 0, 1, 2, 3, \ldots$:

$$c_{2k} = -\frac{\cancel{2}\alpha}{\cancel{2}k} c_{2k-2} = -\frac{\alpha}{k} c_{2k-2}$$

- Now, there are various techniques for solving recurrence rel'ns, but they are a bit outside the scope of this class. Let's look @ the 1st several terms:

$$k=1: \quad c_2 = (-1)\frac{\alpha}{1}\cdot c_0$$

$$k=2: \quad c_4 = (-1)\cdot\frac{\alpha}{2}\cdot(-1)\frac{\alpha}{1}\cdot c_0$$

$$k=3: \quad c_6 = (-1)\frac{\alpha}{3}\cdot(-1)\frac{\alpha}{2}\cdot(-1)\frac{\alpha}{1}\cdot c_0$$

$$\vdots$$

$$c_{2k} = (-1)^k \frac{\alpha^k}{k!} c_0$$

$$\Rightarrow \quad y(x) = c_0 \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^k x^{2k}}{k!} = c_0 e^{-\alpha x^2}$$

Sol'n contains one unknown b/c it was a 1st order ODE

- Let's try one more example. We want to solve the eqn $y'' - x^2 y' + 2y = 0$, subject to the conditions $y(0) = 0$ & $y'(0) = 1$.

$$y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{\sum_{n+2=0}^{\infty}(n+2)(n+1)c_{n+2}x^n} - \underbrace{\sum_{n=0}^{\infty} n\cdot c_n \cdot x^{n+1}}_{\sum_{n-1=0}^{\infty}(n-1)c_{n-1}x^n} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$0 = \sum_{n=-2}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty}(n-1)c_{n-1}x^n + \sum_{n=0}^{\infty} 2c_n x^n$$

$n=-2$ & $n=-1$ terms vanish

First non-zero term @ $n=2$

$$\rightarrow 0 = \underbrace{\sum_{n=0}^{\infty}\Big((n+2)(n+1)c_{n+2} + 2c_n\Big)x^n}_{\text{Write out the 1st two terms in this sum } (n=0 \text{ \& } n=1)\ldots} - \underbrace{\sum_{n=2}^{\infty}(n-1)c_{n-1}x^n}_{\ldots\text{ then combine } n\geq 2 \text{ terms w/ this sum.}}$$

$$= (2\cdot 1\cdot c_2 + 2c_0)x^0 + (3\cdot 2\cdot c_3 + 2c_1)x^1 + \ldots$$

$$\ldots + \sum_{n=2}^{\infty}\Big[(n+2)(n+1)c_{n+2} + 2c_n - (n-1)c_{n-1}\Big]x^n$$

- The 1st two terms (x^0 & x^1) give:

$$c_2 = -c_0 \qquad c_3 = -\frac{1}{3}c_1$$

- The remaining terms satisfy the recurrence rel'n:

$$c_{n+2} = \frac{(n-1)c_{n-1} - 2c_n}{(n+2)(n+1)} \leftarrow n\geq 2$$

So, the 1st few terms are:

$$c_4 = \frac{c_1 - 2c_2}{4\cdot 3} = \frac{1}{12}c_1 - \frac{1}{6}(-c_0) = \frac{1}{6}c_0 + \frac{1}{12}c_1$$

$$c_5 = \frac{2\cdot c_2 - 2c_3}{20} = \frac{1}{10}(-c_0) - \frac{1}{10}\left(-\frac{1}{3}c_1\right) = -\frac{1}{10}c_0 + \frac{1}{30}c_1$$

$$c_6 = \frac{3\cdot c_3 - 2c_4}{30} = \frac{1}{10}c_3 - \frac{1}{15}c_4 = -\frac{1}{30}c_1 - \frac{1}{90}c_0 - \frac{1}{180}c_1$$

$$= -\frac{7}{180}c_1 - \frac{1}{90}c_0$$

- So assuming I haven't made any algebra mistakes (who knows!) the 1st few terms of a sol'n are:

$$y(x) = c_0\cdot\left(1 - x^2 + \frac{1}{6}x^4 - \frac{1}{10}x^5 - \frac{1}{90}x^6 + \ldots\right)$$

$$+ c_1\cdot\left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{7}{180}x^6 + \ldots\right)$$

Two unknowns, as we'd expect for a 2nd order ODE

- Notice, though, that I said a sol'n & not the sol'n. Our problem specified $y(0) = 0$ & $y'(0) = 1$. What does this mean for our series sol'n?

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y(0) = c_0$$ b/c all the other terms vanish @ $x=0$!

$$y(0) = 0 \Rightarrow c_0 = 0 \leftarrow$$ Wipes out half of our sol'n!

$$y'(x) = \sum_{n=0}^{\infty} n \cdot c_n \cdot x^{n-1} = 0 \cdot c_0 \cdot x^{-1} + 1 \cdot c_1 \cdot x^0 + 2 \cdot c_2 \cdot x^1 + \dots$$

$$y'(0) = c_1 = 1$$

DE has many sol'ns. But a specific sol'n requires $y(0) = \dots$, $y'(0) = \dots$, etc.

- Specifying $y(0)$ & $y'(0)$ fixes c_0 & c_1, and since all our c_n are related to c_0 & c_1 by the recurrence rel'ns we have a unique sol'n to our problem!

$$y(x) = x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{7}{180}x^6 + \dots$$

- So this is basically how all series sol'ns work. We assume a series, plug it into the eqn, find the recurrence rel'ns and solve them, then use any conditions to narrow down the sol'n.

- As we said @ the beginning, this approach works for any DE of the form

$$y''(x) + f(x)\,y'(x) + g(x)\,y(x) = h(x)$$

as long as $f, g,$ & h are analytic (can be represented as a power series).

- In later sections we'll look @ other sorts of series sol'ns. But first, we're going to apply what we just learned to an important DE that shows up all over the place in your upper-level physics courses.

LEGENDRE'S EQN & LEGENDRE POLYNOMIALS

- As an application, we're going to study an eqn. that shows up when we try to solve Laplace's eqn in SPC ← (See HW 7!)

$$(1-x^2)\,y''(x) - 2x\,y'(x) + \ell(\ell+1)\,y(x) = 0$$

- Series sol'ns of this eqn will lead us to our 1st class of special functions: <u>LEGENDRE POLYNOMIALS</u>.

- Let's start w/ a similar eqn

Some constant

$$(1-x^2)\,y''(x) - 2x\,y'(x) + \mu\,y(x) = 0$$

- Assume a series sol'n: $y(x) = \sum_{n=0}^{\infty} c_n x^n$

$$\rightarrow 0 = \sum_{n=0}^{\infty} n(n-1)\,c_n \cdot (x^{n-2} - x^n) - 2\sum_{n=0}^{\infty} n \cdot c_n x^n + \mu \sum_{n=0}^{\infty} c_n x^n$$

$$= \underbrace{\sum_{n=0}^{\infty} n(n-1)\,c_n \cdot x^{n-2}}_{n\to n+2} + \sum_{n=0}^{\infty} \left(-n(n-1)c_n - 2n\,c_n + \mu\,c_n\right) x^n$$

$$= \sum_{n=\cancel{-2}\,0}^{\infty} (n+2)(n+1)\,c_{n+2}\,x^n + \sum_{n=0}^{\infty} \left(\mu - n(n+1)\right) c_n x^n$$

$$\hookrightarrow 0 = \sum_{n=0}^{\infty} \left[(n+2)(n+1)\,c_{n+2} + \left(\mu - n(n+1)\right) c_n\right] x^n$$

RECURRENCE REL'N: $$c_{n+2} = -\frac{\left(\mu - n(n+1)\right)}{(n+2)(n+1)}\,c_n$$

- So our series sol'n has c_0 & c_1 undetermined, like the other LINEAR ODEs we've seen, w/ c_n given by the recurrence rel'n $\forall\ n \geq 2$.

- Now, look carefully @ the recurrence rel'n:

$$c_{n+2} = -\frac{(\mu - n(n+1))}{(n+2)(n+1)} c_n \qquad n \geq 0$$

- If μ is just any old #, we'll get two ∞-series sol'ns. But if $\mu = \ell(\ell+1)$, where $\ell \in \mathbb{Z}$ ("ℓ is an integer") something interesting happens:

$$c_{n+2} = -\frac{(\ell(\ell+1) - n(n+1))}{(n+2)(n+1)} c_n$$

Once n reaches ℓ, the numerator vanishes & $c_{n+2} = 0$, $c_{n+4} = 0$, $c_{n+6} = 0$, ...

- If ℓ is an <u>ODD</u> integer, then all the odd terms in the series will vanish for $n > \ell$, giving a <u>FINITE</u> series. The even terms will still be an ∞-series.

- If ℓ is an <u>EVEN</u> integer, it's the opposite.

- If we write out the 1st few terms in the sol'n, we get:

$$y(x) = c_0 \times \left(1 - \frac{\ell(\ell+1)}{2} x^2 + \frac{(\ell+3)(\ell+1)\ell(\ell-2)}{4!} x^4 - \frac{(\ell+5)(\ell+3)(\ell+1)\ell(\ell-2)(\ell-4)}{6!} x^6 + \dots \right)$$

$$+ c_1 \times \left(x - \frac{(\ell+2)(\ell-1)}{3!} x^3 + \frac{(\ell+4)(\ell+2)(\ell-1)(\ell-3)}{5!} x^5 + \dots \right)$$

- So if ℓ is an <u>odd</u> integer the c_1 series stops @ the x^ℓ term, but the c_0 series is infinite.

- And if ℓ is an <u>even</u> integer, the c_0 series stops @ the x^ℓ term but the c_1 series is infinite.

- A <u>finite</u> series always converges, so in either case (ℓ an even or odd integer) @ least one of our sol'ns is a finite polynomial in x. What about the infinite series sol'n?

- To check convergence we can use the <u>RATIO TEST</u>

$$\lim_{n\to\infty}\left|\frac{c_{n+2}\,x^{n+2}}{c_n\,x^n}\right| = \lim_{n\to\infty}\underbrace{\left|\frac{(\ell-n)(\ell+n+1)}{(n+2)(n+1)}\right|}_{\sim\,\frac{-n^2}{n^2}\ \text{if}\ n\gg\ell}|x^2| = |x^2|$$

- So the ∞ series solution only converges for $|x^2|<1$. But it <u>diverges</u> for $|x|\geq 1$! For instance, look @ $\ell=1$:

$$\ell=1:\quad y(x) = c_1 x + c_0\cdot\left(1-x^2-\tfrac{1}{3}x^4-\tfrac{1}{5}x^6-\tfrac{1}{7}x^8-\tfrac{1}{9}x^{10}+\ldots\right)$$

$$\hookrightarrow\ y(\pm1) = \pm c_1 + c_0\cdot\Big(\overset{0}{\cancel{1-1}}\underbrace{-\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{9}+\ldots}_{\text{DIVERGE}}\Big)$$

- Okay, for $\ell\in\mathbb{Z}$ we have two sol'ns: a finite polynomial of order ℓ (which is finite $\forall\, x$) and an ∞ series which converges for $|x|<1$ but <u>not</u> @ $|x|\geq 1$.

- The former are "Legendre Polynomials of the 1st kind" and we denote them by $P_\ell(x)$. The latter are "Legendre Polynomials of the 2nd kind," which we denote $Q_\ell(x)$.

- So a general sol'n of Legendre's equation ($\mu = \ell(\ell+1)$!) is:

$$y(x) = a_1 P_\ell(x) + a_2 Q_\ell(x)$$

- Now there's one small problem. These often show up in sol'ns of Laplace's eqn in SPC, with $x = \cos\theta$. (You'll show this in HW 7!)

- θ is the polar angle, w/ $\theta = 0$ @ the NP & $\theta = \pi$ @ the SP. But $\theta = 0$ is $x = 1$ & $\theta = \pi$ is $x = -1$, so the Q_ℓ all <u>diverge</u> @ the NP & SP!

- For this reason, we usually <u>discard</u> the $Q_\ell(x)$ for most physical applications, <u>unless</u> we aren't considering $x = \pm 1$. We're now going to focus on the $P_\ell(x)$.

<u>RESULT</u>: $\forall$ non-negative $\ell \in \mathbb{Z}$, $\exists$ precisely one sol'n of Legendre's eqn. that converges $\forall$ $|x| \leq 1$.

- The $P_\ell(x)$ are the part of our $y(x)$ sol'n proportional to either c_0 if ℓ is even or c_1 if ℓ is odd. The 1st several are:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

- Here are some properties & important facts that follow from our derivation:

(1) $P_\ell(x)$ is even/odd across $x=0$ for even/odd ℓ. Another (useful) way to state this is

$$P_\ell(-x) = (-1)^\ell P_\ell(x)$$

(2) $P_\ell(x)$ is a polynomial of order ℓ. That is, its largest power is x^ℓ, followed by $x^{\ell-2}$, and so on down to x (ℓ odd) or $x^0 = 1$ (ℓ even).

(3) Since the $P_\ell(x)$ have a finite # of terms they are finite $\forall\, x$. But once x gets large they grow like x^ℓ. We usually only use them on $-1 \le x \le 1$ (or $0 \le \theta \le \pi$ in SPC, w/ $x = \cos\theta$).

(4) The $P_{2n+1}(x)$ are <u>odd</u>, so $P_{2n+1}(0) = 0$.

(5) $P_\ell(1) = 1 \;\forall\, \ell$, even or odd. This means $P_{2n}(-1) = 1$ & $P_{2n+1}(-1) = -1$.

- The 1st few $P_\ell(x)$ are simple, but as ℓ gets bigger there are more terms w/ large coefficients. For example

$$P_{20}(x) = \frac{34{,}461{,}632{,}205}{262{,}144}x^{20} + \ldots - \frac{29{,}113{,}619{,}535}{65{,}536}x^{10} + \ldots$$

- This seems pretty complicated. Do we just let a computer handle it? There is actually a pretty convenient & compact formula for $P_\ell(x)$. It's useful for obtaining a particular $P_\ell(x)$ you've forgotten, and also for working out integrals involving the P_ℓ.

- RODRIGUES' FORMULA is:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}\left((x^2-1)^\ell\right)$$

- We're not going to prove this (though I can post a proof if you'd like!) but let's at least check for one value of ℓ, say $\ell=2$:

$$P_2(x) \stackrel{?}{=} \frac{1}{2^2 2!} \frac{d^2}{dx^2}\left((x^2-1)^2\right) = \frac{1}{8}\frac{d}{dx}\left(2(x^2-1)\,2x\right)$$

$$= \frac{1}{2}\frac{d}{dx}\left(x^3-x\right) = \frac{1}{2}(3x^2-1) \ \checkmark$$

- Now, this isn't the only useful formula for recalling the L.P. It turns out that L.P., like most families of special functions obtained as sol'ns of differential equations, have what is called a <u>GENERATING FUNCTION</u>.

- A generating function is (for our purposes) a function of two variables such that a power series expansion in one of the variables has some set of special functions (of the other variable) as its coefficients. Schematically, it looks like this:

Powers ($\ell \geq 0$) of one variable

$$G(x,h) = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x)$$

L.P. of the other variable

- For Legendre Polynomials of the 1st kind, the generating function is:

$$G(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} \qquad \text{w/ } |h|<1 \ \ \& \ \ |x| \leq 1$$

- It's possible to show that the coefficient of h^ℓ in the power series expansion of $G(x,h)$ satisfies Legendre's eqn. But for our purposes, it's enough to show that the 1st few terms give what we expect.

- Since both $|x| \leq 1$ & $|h| < 1$, $|h^2 - 2xh|$ is always < 1. So:

$$G(x,h) = \frac{1}{\sqrt{1+\lambda}} \quad \text{w/ } \lambda = h^2 - 2xh$$

$$= 1 - \frac{1}{2}\lambda + \frac{3}{8}\lambda^2 - \frac{5}{16}\lambda^3 + \dots$$

Maclaurin series for $1/\sqrt{1+\lambda}$

$$= 1 - \frac{1}{2}(h^2 - 2xh) + \frac{3}{8}(h^2 - 2xh)^2 - \frac{5}{16}(h^2 - 2xh)^3 + \dots$$

$$= 1 - \frac{1}{2}h^2 + xh + \frac{3}{8}(h^4 - 4xh^3 + 4x^2h^2) - \frac{5}{16}(h^6 - 6xh^5 + 12x^2h^4 - 8x^3h^3) + \dots$$

$$= \underset{P_0}{1} + h \cdot \underset{P_1}{x} + h^2 \cdot \underbrace{\left(\frac{3}{2}x^2 - \frac{1}{2}\right)}_{P_2} + h^3 \cdot \underbrace{\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)}_{P_3} + \dots$$

- The generating function is important for many reasons, but we will focus on two of them.

- First, it reveals a number of special properties & identities of L.P. For example:

$$G(1,h) = \frac{1}{\sqrt{1-2h+h^2}} = \frac{1}{\sqrt{(1-h)^2}} = \frac{1}{1-h}$$

$$= 1 + h + h^2 + h^3 + h^4 + \dots$$

$$= P_0(1) + P_1(1)h + P_2(1)h^2 + P_3(1)h^3 + \dots$$

$$\Rightarrow P_\ell(1) = 1 \quad \forall\, \ell \geq 0$$

- Likewise, expanding $G(-1,h)$ as a power series in h confirms that $P_\ell(-1) = (-1)^\ell$

- But we can also perform various manipulations on the series expansion of $G(x,h)$ & its derivatives, which leads to several useful identities satisfied by the $P_\ell(x)$:

1) $\ell P_\ell(x) = (2\ell-1)\, x\, P_{\ell-1}(x) - (\ell-1) P_{\ell-2}(x)$
2) $x \frac{d}{dx} P_\ell(x) = \ell P_\ell(x) + \frac{d}{dx} P_{\ell-1}(x)$
3) $\frac{d}{dx} P_\ell(x) - x \frac{d}{dx} P_{\ell-1}(x) = \ell P_{\ell-1}(x)$
4) $(1-x^2) \frac{d}{dx} P_\ell(x) = \ell P_{\ell-1}(x) - x \ell P_\ell(x)$
5) $(2\ell+1) P_\ell(x) = \frac{d}{dx} P_{\ell+1}(x) - \frac{d}{dx} P_{\ell-1}(x)$
6) $(1-x^2) \frac{d}{dx} P_{\ell-1}(x) = \ell x P_{\ell-1}(x) - \ell P_\ell(x)$

$\vdots$

ALSO REFERRED TO AS "RECURRENCE REL'NS"

- The second reason is that it helps us work out various integrals involving the $P_\ell(x)$. But why would we care about that? Well, I'm glad you asked....

- We've been solving differential eqns by assuming the sol'ns are analytic. And that seems reasonable, because we're doing <u>physics</u>, and physical systems usually* don't do anything mathematically extreme, so it's not a wild assumption to say that a sol'n describing some physical system is probably analytic:

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

*Probably fine, at least for some range of x!

- But if this is a sol'n of an equation that involves Legendre's eqn for different values of ℓ (we'll see why this is next month) shouldn't it also be a combination of $P_\ell(x)$?

- Here's another way to see this. Look @ the 1st few $P_\ell(x)$ & "invert" them to solve for x^0, x^1, x^2, etc...

$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} = \frac{3}{2}x^2 - \frac{1}{2}P_0(x)$

$\rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$

$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x = \frac{5}{2}x^3 - \frac{3}{2}P_1(x) \Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$

- And so on! So our power series for $y(x)$ could also be written as

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$
$$= c_0 \cdot P_0 + c_1 \cdot P_1 + c_2 \cdot \left(\frac{2}{3}P_2 + \frac{1}{3}P_0\right) + c_3 \cdot \left(\frac{2}{5}P_3 + \frac{3}{5}P_1\right) + \dots$$
$$= \left(c_0 + \frac{1}{3}c_2 + \dots\right)P_0(x) + \left(c_1 + \frac{3}{5}c_3 + \dots\right)P_1(x)$$
$$+ \left(\frac{2}{3}c_2 + \dots\right)P_2(x) + \left(\frac{2}{5}c_3 + \dots\right)P_3(x)$$

$$\hookrightarrow y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)$$

- In other words, the $P_\ell(x)$ are a "different basis" for our power series expansion, and we can represent $y(x)$ as some combination of Legendre polynomials. Does this sound familiar?

- This is exactly what we did w/ sines & cosines when we studied Fourier Series. We'll do exactly the same thing here & call it a Legendre series. And just like F.S. we'll use our analogy w/ vectors & dot products to understand how much of each $P_\ell(x)$ is needed for the Legendre series representation of a particular $y(x)$.

- Like $\sin(nx)$ & $\cos(nx)$ on $-\pi \le x \le \pi$, the $P_\ell(x)$ form what we call a "complete, orthogonal set of functions" on $-1 \le x \le 1$. So we can use them to build up a representation of any reasonably behaved function on that interval, just like we can represent a vector by adding up unit vectors in the right proportions.

- We just need to figure out "how much" of each $P_\ell(x)$ is needed for a particular function.

Given $f(x)$, what are these?

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)$$

- This just like a F.S., so is there something like Fourier's Trick? Consider the integral of two $P_\ell(x)$. I claim:

$$\int_{-1}^{1} dx\, P_\ell(x) P_k(x) = \begin{cases} 0 & \text{if } \ell \neq k \\ \text{Const.} & \text{if } \ell = k \end{cases}$$

- In other words, the $P_\ell(x)$ are <u>orthogonal</u> on $-1 \le x \le 1$ just like $\cos(nx)$ & $\sin(nx)$ on $-\pi \le x \le \pi$.

- Is this plausible? True if one of ℓ, k is even & the other is odd, since $P_\ell(x) P_k(x)$ is an odd function of x in that case. And it's true for, say, $\ell=0$ & $k=2$ or $\ell=1$ & $k=3$. But in general?

- Using either the generating function $G(x,h)$, or also by using Legendre's equation, we can prove a lovely identity:

$$\frac{d}{dx}\Big((1-x^2)\big(P_k(x)P_\ell'(x) - P_\ell(x)P_k'(x)\big)\Big) + \big(\ell(\ell+1) - k(k+1)\big) P_\ell(x) P_k(x) = 0$$

$$\rightarrow \underbrace{\int_{-1}^{1} dx \frac{d}{dx}\left((1-x^2)(P_k P_\ell' - P_\ell P_k')\right)}_{(1-x^2)(P_k P_\ell' - P_\ell P_k')\Big|_{-1}^{1} = 0} + \left(\ell(\ell+1) - k(k+1)\right)\int_{-1}^{1} dx\, P_\ell P_k = 0$$

$$\rightarrow \left(\ell(\ell+1) - k(k+1)\right)\int_{-1}^{1} dx\, P_k(x) P_\ell(x) = 0$$

$$\Rightarrow \int_{-1}^{1} dx\, P_k(x) P_\ell(x) = 0 \quad \text{if } \ell \neq k$$

- So we just need to work out the $\ell = k$ case. But first, a <u>very</u> important corollary. Since the $P_\ell(x)$ are finite polynomials, it should be clear that we can rewrite any <u>FINITE</u> series of order N in terms of $P_0, P_1, P_2, \ldots, P_N$.

$$\sum_{k=0}^{N} c_k x^k = \sum_{k=0}^{N} b_k P_k(x)$$

Just by expressing the x^k as combinations of Legendre polynomials; $1 = P_0$, $x = P_1$, etc...

So if $N < \ell$, it follows that

$$\int_{-1}^{1} dx\, P_\ell(x) \times \left(\text{Any polynomial of order } N\right) = 0$$

- Now what about the $\ell = k$ case? Using the recurrence relation

$$\ell\, P_\ell(x) = x \frac{d}{dx} P_\ell(x) - \frac{d}{dx} P_{\ell-1}(x)$$

for one of the factors, and after some integration-by-parts:

$$\int_{-1}^{1} dx\, P_\ell(x)^2 = \frac{2}{2\ell+1}$$

Check: $\int_{-1}^{1} dx\, P_1^2 = \int_{-1}^{1} dx\, x^2 = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3}$

$$\Longrightarrow \int_{-1}^{1} dx\, P_\ell(x) P_k(x) = \begin{cases} 0 & \ell \neq k \\ \dfrac{2}{2\ell+1} & \ell = k \end{cases}$$

- This integral is our "inner product" for the $P_\ell(x)$, and we'll use it the same way we used a similar integral for sines & cosines!

- That is, suppose you have some function $f(x)$ on $-1 \le x \le 1$, and you want to write it as a <u>LEGENDRE</u> <u>SERIES</u>.

$$f(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x)$$

How do we get these? How much of each $P_\ell(x)$?

- To find the coefficient c_k of the $\ell = k$ term, multiply both sides by $P_k(x)$:

$$P_k(x) f(x) = \sum_{\ell=0}^{\infty} c_\ell P_k(x) P_\ell(x)$$

THIS IS <u>FOURIER'S</u> <u>TRICK</u> FOR LEGENDRE SERIES!

And now integrate from -1 to 1:

$$\int_{-1}^{1} dx\, P_k(x) f(x) = \sum_{\ell=0}^{\infty} c_\ell \underbrace{\int_{-1}^{1} dx\, P_k(x) P_\ell(x)}_{\frac{2}{2k+1} \text{ if } \ell = k, \text{ otherwise } 0}$$

$$= \frac{2}{2k+1} c_k$$

All the $\ell \ne k$ terms multiplied by zero!

$$\Rightarrow \boxed{c_k = \frac{2k+1}{2} \int_{-1}^{1} dx\, P_k(x) f(x)}$$

This is <u>exactly</u> what we did w/ Fourier Series, we're just using a different basis – Legendre Polynomials instead of sines & cosines

- Why would we want to do this? Recall how we introduced F.S. – we knew sine waves were the natural modes of a vibrating string, and suspected we could use them to build up more realistic (but more complicated) motions of a string that result from plucking or striking it.
- The same is true here! Legendre polynomials are the simple & natural solutions of Legendre's equation, and we use them as building blocks of more complicated solutions! You'll use this throughout E&M and QM.

<u>Ex</u> $f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 < x \le 1 \end{cases}$

$$c_0 = \frac{2\cdot 0+1}{2}\int_{-1}^{1} dx\, P_0(x)\cdot f(x) = \frac{1}{2}\left(\int_{-1}^{0} dx\, 1\cdot(-1) + \int_{0}^{1} dx\, 1\cdot 1\right)$$

$$= \frac{1}{2}\left(-x\Big|_{-1}^{0} + x\Big|_{0}^{1}\right) = \frac{1}{2}\left(0-1+1-0\right) = 0$$

$$c_1 = \frac{2\cdot 1+1}{2}\int_{-1}^{1} dx\, P_1(x)\cdot f(x) = \frac{3}{2}\left(\int_{-1}^{0} dx\,(-x) + \int_{0}^{1} dx\, x\right)$$

$$= \frac{3}{2}\cdot\left(-\frac{1}{2}x^2\Big|_{-1}^{0} + \frac{1}{2}x^2\Big|_{0}^{1}\right) = \frac{3}{2}\cdot\left(-0+\frac{1}{2}+\frac{1}{2}-0\right) = \frac{3}{2}$$

$c_2 = 0 \leftarrow$ WHY? ($P_2(x)$ even, $f(x)$ odd!)

$$c_3 = \frac{2\cdot 3+1}{2}\int_{-1}^{1} dx\, P_3(x) f(x) = \frac{7}{2}\cdot\left(\int_{-1}^{0} dx\,(-1)\left(\frac{5}{2}x^3-\frac{3}{2}x\right) + \int_{0}^{1} dx\left(\frac{5}{2}x^3-\frac{3}{2}x\right)\right)$$

$$= \frac{7}{2}\cdot\left(-\left(\frac{5}{8}x^4-\frac{3}{4}x^2\right)\Big|_{-1}^{0} + \left(\frac{5}{8}x^4-\frac{3}{4}x^2\right)\Big|_{0}^{1}\right)$$

$$= \frac{7}{2}\cdot\left(-0+\frac{5}{8}-\frac{3}{4}+\frac{5}{8}-\frac{3}{4}-0\right) = \frac{7}{2}\cdot\left(\frac{10}{8}-\frac{\cancel{6}^{\,12}}{\cancel{4}\,8}\right) = -\frac{7}{8}$$

$$\hookrightarrow f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \frac{75}{128}P_7(x) + \ldots$$

The Legendre Series representation of $f(x)$

- A few important points. First, the conditions for $f(x)$ to have a Legendre Series representation are essentially the same as the Dirichlet conditions for the Fourier Series.

- Second, based on our earlier result for integrals of the $P_\ell(x)$, the L.S. for a polynomial of order k <u>always</u> stops w/ the $\ell = k$ term:

$c_\ell = 0 \ \forall \ \ell \geq 25!$

$$f(x) = 17x^{24} - 32x^{23} + \dots + 7x + 90 = c_{24}P_{24}(x) + c_{23}P_{23}(x) + \dots + c_0 P_0(x)$$

- So for a <u>simple</u> polynomial, finding the L.S. may be easier or quicker to do algebraically. For example:

$$f(x) = 2x^2 - 4x + 7 = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)$$
$$= c_0 + c_1 x + \tfrac{3}{2}c_2 x^2 - \tfrac{1}{2}c_2$$

$$\hookrightarrow 2 = \tfrac{3}{2}c_2 \qquad -4 = c_1 \qquad 7 = c_0 - \tfrac{1}{2}c_2$$

$$\Rightarrow c_2 = \tfrac{4}{3} \qquad c_1 = -4 \qquad c_0 = \tfrac{23}{3}$$

- But in general, we can always find the L.S. coefficients via

$$\underbrace{c_\ell}_{\text{Coeff. of "}\vec{E}_\ell\text{"}} = \underbrace{\frac{2\ell+1}{2}}_{\text{B/c "}\vec{E}_\ell\cdot\vec{E}_\ell = \frac{2}{2\ell+1}\text{"}} \underbrace{\int_{-1}^{1} dx\, P_\ell(x)\, f(x)}_{\text{Analogous to } \vec{E}_\ell\cdot\vec{A} \text{ in our vector example}}$$

Sometimes you can do these integrals for general ℓ, sometimes just the 1st few values of ℓ.

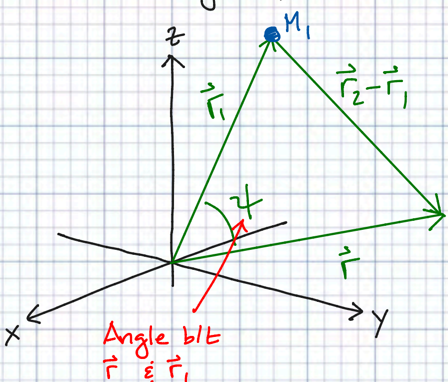
- We often work in SPC where our variable is $x = \cos\theta$. Then $dx = -\sin\theta\, d\theta$, $x = 1 \Rightarrow \theta = 0$, $x = -1 \Rightarrow \theta = \pi$, and

$$\int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta) P_k(\cos\theta) = \begin{cases} 0, & \ell \neq k \\ \frac{2}{2\ell+1}, & \ell = k \end{cases} \quad \Rightarrow c_\ell = \frac{2\ell+1}{2}\int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta)\, F(\theta)$$

- One last point. The generating function for the $P_\ell(x)$ was

$$G(x,h) = \frac{1}{\sqrt{1-2xh+h^2}}$$

This comes up all the time when we have a $1/r$ potential, like in gravity or E & M.



$$V_{grav}(\vec{r}) = -G_N \frac{M_1}{|\vec{r} - \vec{r}_1|}$$

$|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$

$$|\vec{r} - \vec{r}_1| = \sqrt{(\vec{r}-\vec{r}_1)\cdot(\vec{r}-\vec{r}_1)}$$

$$= \sqrt{\vec{r}\cdot\vec{r} + \vec{r}_1\cdot\vec{r}_1 - 2\vec{r}\cdot\vec{r}_1}$$

$$= \sqrt{(r)^2 + (r_1)^2 - 2 r r_1 \underbrace{\cos\psi}}$$

Dist. from origin of pt. w/ pos. vector $\vec{r}$

r_1 is dist. of M_1 from origin

$\measuredangle$ b/t $\vec{r}$ & $\vec{r}_1$

So if $r > r_1$, then

$$V_{grav} = -\frac{G_N M_1}{r\sqrt{1 - 2\frac{r_1}{r}\cos\psi + \left(\frac{r_1}{r}\right)^2}}$$

$x = \cos\psi$ & $h = r_2/r_1$!

$$= -\frac{G_N M_1}{r} \times \sum_{\ell=0}^{\infty} \left(\frac{r_1}{r}\right)^\ell P_\ell(\cos\psi)$$

- The P_ℓ naturally show up if we expand V_{grav} (or $V_{Coulomb}$) in powers of r/r_1. This isn't a coincidence, since both satisfy equations of the form $\nabla^2 V = 0$ away from M_1 (or q_1)!

The Method of Frobenius

- We've successfully applied our ∞-series technique to several eqns now. It's powerful & useful & you're going to see it a lot.

- But sometimes you get an eqn where a Maclaurin series just doesn't work. It may give nonsense, or an apparently trivial ($y=0$) sol'n.

- For example, consider

$$x^2 y'' + 4xy' + (x^2+2)y = 0$$

$$y \stackrel{?}{=} \sum_{n=0}^{\infty} c_n x^n$$

Re-index sums, etc..

$$0 = \underbrace{2c_0}_{\text{So } c_0=0?} + \underbrace{6c_1}_{\text{And } c_1=0?} x + \sum_{n=2}^{\infty} \underbrace{\left[(n+2)(n+1)c_n + c_{n-2}\right]}_{\text{And therefore all the remaining } c_n = 0?} x^n$$

- It looks like the sol'n has to be $y=0$, right? No, that can't be right! If x was really close to 0, so that $x^2 y \ll 2y$ in the last term, wouldn't we have

$$x^2 y'' + 4xy' + 2y \simeq 0$$

Try $y \sim x^\alpha \Rightarrow \alpha = -1$ or -2

- The problem was assuming a series sol'n that started w/ a constant. Instead, let's try a Generalized Power Series:

$$y(x) = \underbrace{x^s}_{\substack{\uparrow \\ \text{Eqn will determine } s}} \sum_{n=0}^{\infty} c_n x^n \leftarrow c_0 x^s + c_1 x^{s+1} + c_2 x^{s+2} + \ldots$$

- If we repeat our series method we arrive at:

$$0 = \sum_{n=0}^{\infty} (n+s+2)(n+s+1)\, c_n x^{n+s} + \sum_{n=2}^{\infty} c_{n-2} x^{n+s}$$

$$= \underbrace{(s+2)(s+1)}\, c_0 + (s+3)(s+2)\, c_1 + \sum_{n=2}^{\infty} \left[(n+s+2)(n+s+1)\, c_n + c_{n-2}\right] x^{n+s}$$

'INDICIAL EQUATION': $(s+2)(s+1) = 0$

- We determine s by setting the 1st term to zero – this is the INDICIAL EQUATION. Here we get $s=-2$ or $s=-1$, so we get two sol'ns (as expected).

- Let's look @ the $s=1$ sol'n first. If $s=-1$ then

$$0 = \underbrace{0 \cdot c_0}_{c_0 \text{ undet.}} + \underbrace{2 \cdot 1 \cdot c_1}_{c_1 = 0} + \sum_{n=2}^{\infty} \underbrace{\left[(n+1)n \cdot c_n + c_{n-2}\right]}_{c_n = -\frac{c_{n-2}}{n(n+1)}} x^{n-2}$$

$$c_n = -\frac{c_{n-2}}{n(n+1)} \rightarrow c_2 = -\frac{c_0}{3!}, \quad c_4 = -\frac{c_2}{5 \cdot 4} = (-1)^2 \frac{c_0}{5!}$$

$$c_6 = -\frac{c_4}{7 \cdot 6} = (-1)^3 \frac{c_0}{7!}, \ldots, \quad c_{2k} = (-1)^k \frac{c_0}{(2k+1)!}$$

$$\rightarrow y(x) = \frac{1}{x} \sum_{n=0}^{\infty} c_0 \cdot (-1)^k \frac{x^{2k}}{(2k+1)!} = c_0 \cdot \frac{1}{x} \sum_{n=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!}$$

If this were $2k+1$ we'd have $\sin(x)$!

$$\Rightarrow y(x) = \frac{c_0}{x^2} \sin(x) \text{ is the } s=-1 \text{ sol'n}$$

- Now check the $s=2$ solution. If $s=-2$ then

$$0 = \underbrace{0 \cdot c_0 + 0 \cdot c_1}_{c_0 \,\&\, c_1 \text{ both undetermined}} + \sum_{n=2}^{\infty} \underbrace{\left[n \cdot (n-1)\, c_n + c_{n-2}\right]}_{c_n = -\frac{c_{n-2}}{n(n-1)}} x^{n-2}$$

$$\rightarrow c_2 = -\frac{c_0}{2!}, \quad c_4 = -\frac{c_2}{12} = \frac{c_0}{4!}, \ldots \qquad c_3 = -\frac{1}{3!} c_1, \quad c_5 = -\frac{c_3}{20} = +\frac{c_1}{5!}, \ldots$$

- So the $s=-2$ solution is

$$y(x) = c_0 \frac{1}{x^2} \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}_{\cos x} + c_1 \frac{1}{x^2} \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}_{\sin x}$$

$$\Rightarrow y(x) = c_0 \frac{\cos x}{x^2} + c_1 \frac{\sin x}{x^2}$$

In this case, $s=-2$ sol'n includes our $s=-1$ sol'n. Not always the case!

- There were two possibilities — $s=-1$ or $s=-2$ — and the $s=-1$ case was included in the $s=-2$ case. So the general sol'n of this eqn. is

$$y(x) = c_0 \frac{\cos x}{x^2} + c_1 \frac{\sin x}{x^2}$$

Two unknowns, as expected!

- This approach is called the "METHOD OF FROBENIUS." It usually gives you generalized power series sol'ns.
- Wait... usually? When does it work?
- A 2nd order ODE will have a quadratic Indicial eqn w/ two roots s_1 & s_2. If both are real & $s_2 - s_1$ is not an integer (including zero!) then we get 2 generalized power series solutions.
- But if $s_2 - s_1 = 0$ — a double root of the indicial eqn — or sometimes when $s_2 - s_1 \in \mathbb{Z}$, we get a funny sort of solution. In those cases the sol'ns may be

$$y(x) = y_1(x) \log x + y_2(x)$$

where $y_1(x)$ & $y_2(x)$ are both generalized power series. This is called FUCHS'S THEOREM.