Review of Series

▨ TAYLOR, MACLAURIN, & OTHER ∞ SERIES

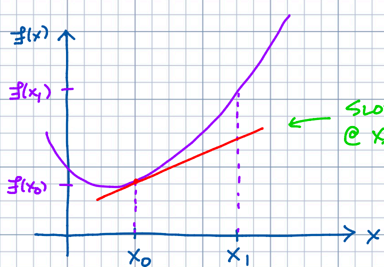
- Over the next several weeks we're going to learn about FOURIER SERIES.

- These are ∞ series built from sines & cosines that we use to represent periodic functions, or functions on a finite interval (like a violin string) that we know to be made of standing waves.

- First, though, let's review some relevant facts about the ∞ series we're already familiar with.

- Suppose you know all about a function $f(x)$ @ some pt. x_0. You know it's value, its 1^{st} derivative, its 2^{nd} derivative, & so on.

- What value does it take @ $x = x_1$, some other point?



So $f(x_0) + f'(x_0) \cdot (x_1 - x_0)$ is $\simeq$ right if x_1 is really close, but not exactly right! And could get wore as x_1 gets further away.

- SOL'N: Add corrections. Get TAYLOR SERIES

$$f(x_1) = f(x_0) + f'(x_0) \cdot (x_1 - x_0) + \frac{1}{2} f''(x_0) \cdot (x_1 - x_0)^2 + \ldots$$

$$\Rightarrow f(x_1) = f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x_0} \cdot (x_1 - x_0)^n$$

- Example: $e^2 = 7.389$. What is $e^{2.1}$?

$\frac{d^n e^x}{dx^n} = e^x$!

$f(x) = e^x \quad f'(x) = e^x \quad f''(x) = e^x \quad \ldots$

$\hookrightarrow f(x_1) = f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(x_0) \cdot (x_1 - x_0)^n$

$$e^{2.1} = e^2 + \sum_{n=1}^{\infty} \frac{1}{n!} e^2 \cdot (2.1 - 2)^n$$

$$= \underset{7.389}{e^2} \times \Big(\underbrace{1 + \sum_{n=1}^{\infty} \frac{1}{n!} (0.1)^n}_{1 + 0.1 + \frac{1}{2}(0.1)^2 + \frac{1}{6}(0.1)^3 + \frac{1}{24}(0.1)^4 + \ldots = 1.105} \Big)$$

DEFINITELY CONVERGES

$\hookrightarrow e^{2.1} = 7.389 \times 1.105 = 8.166$ ✓

- If we use the full ∞ series it converges to the <u>exact</u> value of $e^{2.1}$; if we only evaluate a <u>finite</u> number of terms then it is approximate.

$$e^{2.1} \simeq e^2 \times \left(1 + 0.1 + \frac{1}{2}(0.1)^2 \right) = 8.165$$

ALMOST!

- Example: $\cos(\pi/4) = 0.707$. $\cos(\pi/3) = ?$

$f(x) = \cos(x)$

$f'(x) = -\sin(x) \Rightarrow f'(\pi/4) = -0.707$

$f''(x) = -\cos(x) \Rightarrow f''(\pi/4) = -0.707$

$f^{(3)}(x) = \sin(x) \Rightarrow f^{(3)}(\pi/4) = 0.707$

$f^{(4)}(x) = \cos(x) \Rightarrow f^{(4)}(\pi/4) = 0.707$

$\vdots$ (REPEATS)

$\hookrightarrow \cos\left(\frac{\pi}{3}\right) \simeq 0.707 + (-0.707) \times \left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{1}{2}(-0.707) \times \left(\frac{\pi}{3} - \frac{\pi}{4}\right)^2 + \ldots$

$= 0.4999$ w/ up to $n = 3$ in sum!

- When $x_0 = 0$, we call this a Maclaurin series.

- Example: Maclaurin series for $(1-x)^{-1}$?

$$f(x) = \frac{1}{1-x} \Rightarrow f(0) = 1$$

$$f'(x) = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = 1$$

$$f''(x) = -2 \cdot \frac{1}{(1-x)^3} \cdot (-1) = \frac{2}{(1-x)^3} \Rightarrow f''(0) = 2$$

$$f'''(x) = -3 \cdot \frac{2}{(1-x)^4} \cdot (-1) = \frac{6}{(1-x)^4} \Rightarrow f^{(3)}(0) = 6$$

$$\vdots$$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \Rightarrow f^{(n)}(0) = n!$$

$$\hookrightarrow \frac{1}{1-x} = 1 + x + \frac{1}{2} \cdot 2x^2 + \frac{1}{6} \cdot 6x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

- REMEMBER: This does not converge $\forall x$! Only for $|x| < 1$, or $-1 < x < 1$. Easy to see problem when $x \geq 1$ – all terms get bigger so can never converge. The function is perfectly okay for $x > 1$ or $x \leq -1$; the series just doesn't work there!

- Once we know a few Maclaurin series we can re-use them.

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} n\, x^{n-1}$$

This series has same rad. of convergence, but derivatives usually worsen convergence b/c summand is larger!

$$\int dx \frac{1}{1+x} = \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+n} x^{n+1}$$

ln(1+x) has a Maclaurin series, but ln(x) does not. Not all functions do!

- So, you know how to write many functions as infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \underline{\text{or}} \quad f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left.\frac{d^n f(x)}{dx^n}\right|_{x_0} (x-x_0)^n$$

- In general, these series converge only for some range of values of x — the radius of convergence.

- Now we'll do something similar, but instead of x^n we'll build our series out of functions like $\cos(x)$ or $\sin(17x)$.

- These 'FOURIER SERIES' will be especially useful when we need to represent either a periodic function, or a function on a finite interval that we expect is related to periodic functions (like standing waves).