Special Relativity

Spacetime Coordinates:

$$x^\mu = (c\,t, \vec{x}) \qquad x_\mu = (c\,t, -\vec{x}) \qquad \vec{x} = (x^1, x^2, x^3) = (x, y, z)$$

4-Momentum:

$$p^\mu = (E/c, \vec{p}) \qquad p_\mu = (E/c, -\vec{p}) \qquad \vec{p} = (p^1, p^2, p^3) = (p_x, p_y, p_z)$$

Derivatives:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) \qquad \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) \qquad \vec{\nabla} = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Metric:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \eta^\mu{}_\nu = \delta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Raising and Lowering Indices:

$$a^\mu = \eta^{\mu\nu}\, a_\nu \qquad a_\mu = \eta_{\mu\nu}\, a^\nu$$

Einstein Summation Convention:

$$a^\mu\, b_\mu = a^0\, b_0 + a^1\, b_1 + a^2\, b_2 + a^3\, b_3 = a^0\, b^0 - \vec{a}\cdot\vec{b}$$

Lorentz Transformations:

$$a'_\mu = \Lambda_\mu{}^\nu\, a_\nu \qquad a'^\mu = \Lambda^\mu{}_\nu\, a^\nu$$
$$\eta^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\, \eta^{\alpha\beta} \qquad \eta_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta\, \eta_{\alpha\beta}$$

Properties of Lorentz Transformations:

1. $\eta^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\, \eta^{\alpha\beta}$
2. $|\det \Lambda^\mu{}_\nu| = 1$
3. $(\Lambda_1)^\mu{}_\alpha\, (\Lambda_2)^\alpha{}_\nu = (\Lambda_{12})^\mu{}_\nu$
4. $|\Lambda^0{}_0| \geq 1$

Boost Along the z-Axis:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\,\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\,\gamma & 0 & 0 & \gamma \end{pmatrix} \qquad \text{with} \quad \beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu = \big(\gamma\,(c\,t - \beta\,z),\, x,\, y,\, \gamma\,(z - \beta\,c\,t)\big)$$
$$p'^\mu = \Lambda^\mu{}_\nu\, p^\nu = \big(\gamma\,(E/c - \beta\,p_z),\, p_x,\, p_y,\, \gamma\,(p_z - \beta\,E/c)\big)$$

Rotation About the y-Axis:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} = \big(c\,t,\; x\,\cos\theta - z\,\sin\theta,\; y,\; z\,\cos\theta + x\,\sin\theta\big)$$

$$p'^{\mu} = \Lambda^{\mu}{}_{\nu}\, p^{\nu} = \big(E/c,\; p_x\,\cos\theta - p_z\,\sin\theta,\; p_y,\; p_z\,\cos\theta + p_x\,\sin\theta\big)$$

Relativistic Kinematics

Conservation of 4-Momentum in Any Process:

$$\sum_{i=\text{initial}} p_i^{\mu} = \sum_{j=\text{final}} p_j^{\mu}$$

Massive Particles:

$$p^{\mu}\, p_{\mu} = \frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = m^2\, c^2$$

Massless Particles:

$$p^{\mu}\, p_{\mu} = 0 \qquad\qquad p^{\mu} = |\vec{p}|\,(\,1,\, \vec{\varepsilon}\,) \quad \text{with } \vec{\varepsilon}\cdot\vec{\varepsilon} = 1$$

Important Matrices

2×2 Matrices are written out explicitly, and 4×4 matrices are written out in terms of 2×2 blocks. The 2×2 identity matrix is

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Pauli Matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^i\,\sigma^j = \delta^{ij}\,\mathbb{1} + i\,\epsilon^{ijk}\,\sigma^k \qquad [\,\sigma^i, \sigma^j\,] = 2\,i\,\epsilon^{ijk}\,\sigma^k \qquad \{\,\sigma^i, \sigma^j\,\} = 2\,\delta^{ij}$$

Dirac Matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\{\,\gamma^{\mu}, \gamma^{\nu}\,\} = 2\,\eta^{\mu\nu}$$

$$\not{a} = a_{\mu}\,\gamma^{\mu} = a^0\,\gamma^0 - a^1\,\gamma^1 - a^2\,\gamma^2 - a^3\,\gamma^3$$

Relativistic Wave Equations

Klein-Gordon Equation (Spin-0 Particle):

$$\big(\partial^\mu \partial_\mu + m\big)\phi = 0 \qquad\qquad \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^{\,2}$$

Dirac Equation:

$$(i\,\gamma^\mu\,\partial_\mu - m)\;\psi = 0$$

Maxwell's Equations (Massless Spin-1 Particle):

$$\partial_\mu\,F^{\mu\nu} = 0 \qquad \text{with} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu F^{\mu\nu} = 0 \quad \Rightarrow \quad \begin{cases} \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = 0 \end{cases}$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \quad \Rightarrow \quad \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{cases}$$

Solutions of the Dirac Equation

Some terms in these equations include implicit factors of the 2×2 or 4×4 identity matrices.

Plane-wave solutions:

$$\psi(x) = \begin{cases} a\,e^{-\frac{i}{\hbar}\,p_\mu\,x^\mu}\;u(p) & \text{(fermions)} \\ a\,e^{\frac{i}{\hbar}\,p_\mu\,x^\mu}\;v(p) & \text{(anti-fermions)} \end{cases}$$

Dirac Spinors:

$$(\not{p} - m\,c)\,u^{(s)}(p) = 0 \qquad (\not{p} + m\,c)\,v^{(s)}(p) = 0$$

$$\bar{u}^{(s)}(p)\,(\not{p} - m\,c) = 0 \qquad \bar{v}^{(s)}(p)\,(\not{p} + m\,c) = 0$$

Conjugate Spinors:

$$\bar{u} = u^\dagger\,\gamma^0 \qquad \bar{v} = v^\dagger\,\gamma^0$$

$$\bar{\psi} = \psi^\dagger\,\gamma^0$$

Conjugate Matrices:

$$\bar{A} = \gamma^0\,A^\dagger\,\gamma^0 \qquad \text{for any } 4\times 4 \text{ matrix } A.$$

Complete Set of Plane-wave Solutions:

$$u^{(1)}(p) = N\begin{pmatrix} 1 \\ 0 \\ \frac{c\,p_z}{E+m\,c^2} \\ \frac{c\,(p_x+i\,p_y)}{E+m\,c^2} \end{pmatrix} \qquad u^{(2)}(p) = N\begin{pmatrix} 0 \\ 1 \\ \frac{c\,(p_x-i\,p_y)}{E+m\,c^2} \\ \frac{-c\,p_z}{E+m\,c^2} \end{pmatrix}$$

$$v^{(1)}(p) = N\begin{pmatrix} \frac{c\,(p_x-i\,p_y)}{E+m\,c^2} \\ \frac{-c\,p_z}{E+m\,c^2} \\ 0 \\ 1 \end{pmatrix} \qquad v^{(2)}(p) = -N\begin{pmatrix} \frac{c\,p_z}{E+m\,c^2} \\ \frac{c\,(p_x+i\,p_y)}{E+m\,c^2} \\ 1 \\ 0 \end{pmatrix} \qquad \text{with } N = \sqrt{\frac{E+m\,c^2}{c}}$$

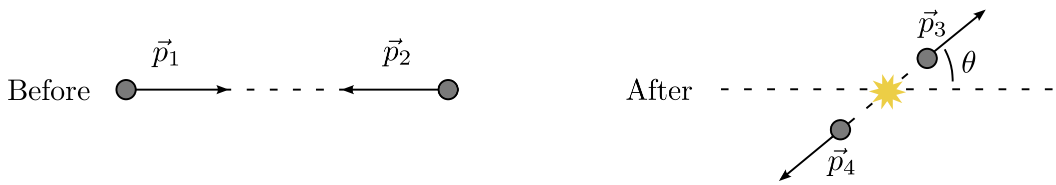
Fermi's Golden Rule

$1 \to 2$ Decays in the Rest Frame of the Decaying Particle:

$$\Gamma = \frac{\mathcal{S}\,|\vec{p}|}{8\pi\hbar\, {m_1}^2\, c}\,|\mathcal{M}|^2$$

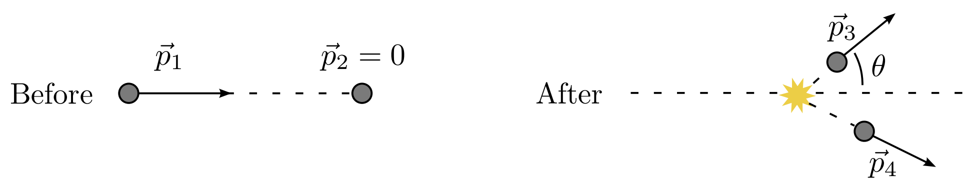
$2 \to 2$ Scattering in the CM Frame:

$$\left.\frac{d\,\sigma}{d\,\Omega}\right|_{CM} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\mathcal{S}\,|\mathcal{M}|^2}{(E_1+E_2)^2}\,\frac{|\vec{p}_f|}{|\vec{p}_i|} \qquad \text{with } |\vec{p}_i| = |\vec{p}_1| = |\vec{p}_2| \text{ and } |\vec{p}_f| = |\vec{p}_3| = |\vec{p}_4|$$



$2 \to 2$ Elastic Scattering in the Lab Frame ($m_3 = m_1$, $m_4 = m_2$):

$$\left.\frac{d\,\sigma}{d\,\Omega}\right|_{\text{Lab}} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{|\vec{p}_3|^2\,\mathcal{S}\,|\mathcal{M}|^2}{m_2\,|\vec{p}_1|\,\left|(E_1 + m_2\,c^2)\,|\vec{p}_3| - |\vec{p}_1|\,E_3\,\cos\theta\right|}$$



Particle Data

Mass in MeV/c^2, liftime in seconds, charge in units of the proton charge.

Leptons (Spin 1/2)

Generation	Flavor	Charge	Mass	Lifetime
First	e	-1	0.510999	∞
	ν_e	0	~ 0	∞
Second	μ	-1	105.659	2.197×10^{-6}
	ν_μ	0	~ 0	∞
Third	τ	-1	1776.88	2.91×10^{-13}
	ν_τ	0	~ 0	∞

Quarks (Spin 1/2)

Generation	Flavor	Charge	Mass	Other Quantum Numbers
First	d	-1/3	7	0
	u	2/3	3	0
Second	s	-1/3	120	Strangeness = -1
	c	2/3	1200	Charm = +1
Third	b	-1/3	4300	Bottomness = -1
	t	2/3	174000	Topness = +1

(Note: Quark masses are approximate.)

Baryons (Spin 1/2)

Baryon	Quark Content	Charge	Mass	Lifetime
p	uud	1	938.272	∞
n	udd	0	939.565	885.7
Λ	uds	0	1115.68	2.63×10^{-10}
Σ^+	uus	1	1189.37	8.02×10^{-11}
Σ^0	uds	0	1192.64	7.4×10^{-20}
Σ^-	dds	-1	1197.45	1.48×10^{-10}
Ξ^0	uss	0	1314.8	2.90×10^{-10}
Ξ^-	dss	-1	1321.3	1.64×10^{-10}
Λ_c^+	udc	1	2286.5	2.00×10^{-13}

Baryons (Spin 3/2)

Baryon	Quark Content	Charge	Mass	Lifetime
Δ	uuu, uud, udd, ddd	2, 1, 0, -1	1232	5.6×10^{-24}
Σ^*	uus, uds, dds	1, 0, -1	1385	1.8×10^{-23}
Ξ^*	uss, dss	0, -1	1533	6.9×10^{-23}
Ω^-	sss	-1	1672	8.2×10^{-11}

Vector Mesons (Spin 1)

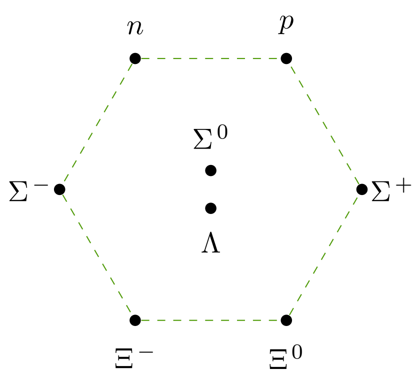
Meson	Quark Content	Charge	Mass	Lifetime
ρ	$u\,\bar{d}$, $(u\,\bar{u} - d\,\bar{d})/\sqrt{2}$, $d\,\bar{u}$	1, 0, 0, -1	775.5	4×10^{-24}
K^*	$u\,\bar{s}$, $d\,\bar{s}$, $d\,\bar{d}$, $s\,\bar{u}$	1, 0, 0, -1	894	1×10^{-23}
ω	$(u\,\bar{u} + d\,\bar{d})/\sqrt{2}$	0	782.6	8×10^{-23}
ψ	$c\,\bar{c}$	0	3097	7×10^{-21}
D^*	$c\,\bar{d}$, $c\,\bar{u}$, $u\,\bar{c}$, $d\,\bar{c}$	1, 0, 0, -1	2008	3×10^{-21}
Υ	$b\,\bar{b}$	0	9460	1×10^{-20}

Pseudoscalar Mesons (Spin 0)

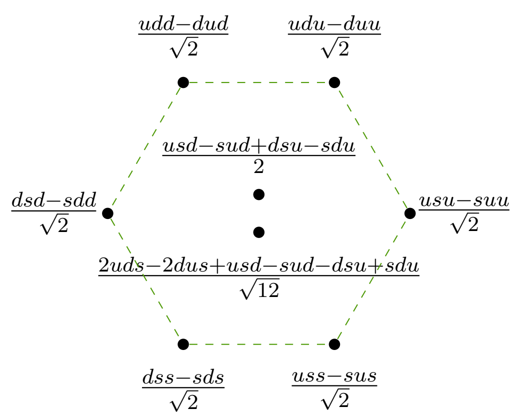
Meson	Quark Content	Charge	Mass	Lifetime
$\pi^{\pm}$	$u\,\bar{d},\ d\,\bar{u}$	±1	139.570	2.6×10^{-8}
π^{0}	$(u\,\bar{u}-d\,\bar{d})/\sqrt{2}$	0	134.977	8.4×10^{-17}
$K^{\pm}$	$u\,\bar{s},\ s\,\bar{u}$	±1	493.68	1.24×10^{-8}
$K^{0},\ \bar{K}^{0}$	$d\,\bar{s},\ s\,\bar{d}$	0	497.65	$K_S^0:\ 8.95\times10^{-11}$ $K_L^0:\ 5.11\times10^{-8}$
η	$(u\,\bar{u}+d\,\bar{d}-2\,s\,\bar{s})/\sqrt{6}$	0	547.51	5.1×10^{-19}
η'	$(u\,\bar{u}+d\,\bar{d}+s\,\bar{s})/\sqrt{3}$	0	957.78	3.2×10^{-21}
$D^{\pm}$	$c\,\bar{d},\ d\,\bar{c}$	±1	1869.3	1.04×10^{-12}
$D^{0},\ \bar{D}^{0}$	$c\,\bar{u},\ u\,\bar{c}$	0	1864.5	4.1×10^{-13}
$D_s^{\pm}$	$c\,\bar{s},\ s\,\bar{c}$	±1	1968.2	5.0×10^{-13}
$B^{\pm}$	$u\,\bar{b},\ b\,\bar{u}$	±1	5279.0	1.6×10^{-12}
$B^{0},\ \bar{B}^{0}$	$d\,\bar{b},\ b\,\bar{d}$	0	5279.4	1.5×10^{-12}

Baryons, Mesons, and the Eightfold Way

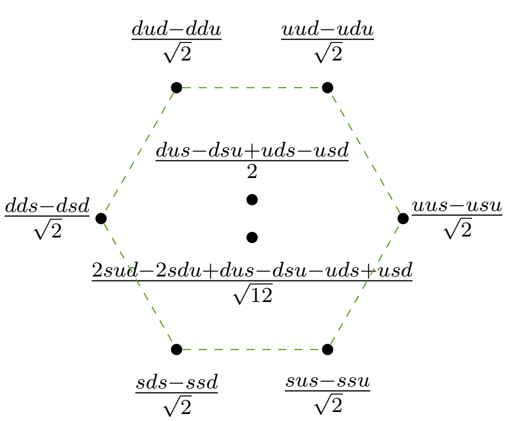
Light Baryons ($\ell = 0$)



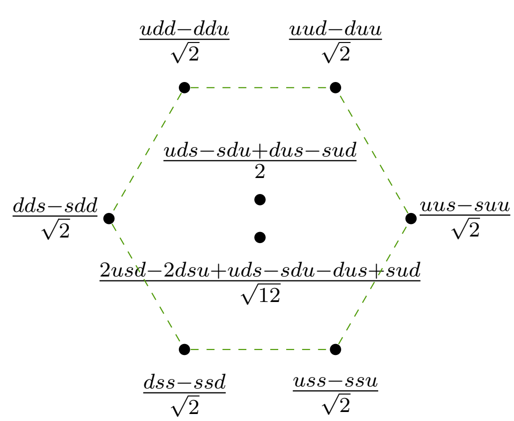
The Baryon Octet



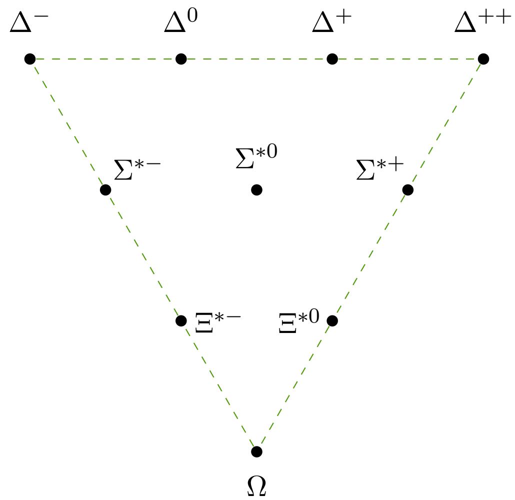
χ_{12} : Antisymmetric in 1 and 2



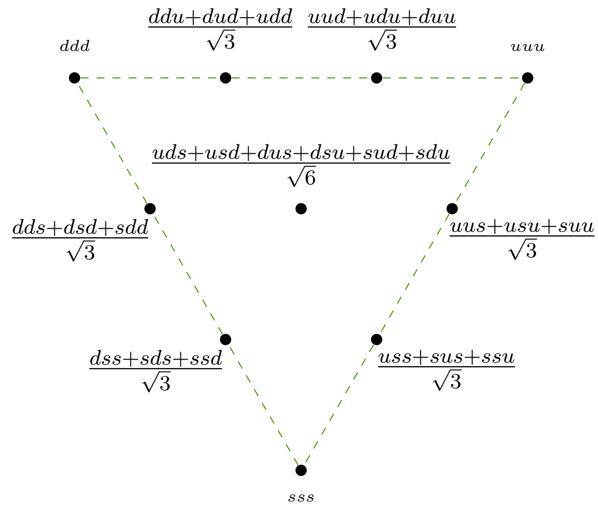
χ_{23} : Antisymmetric in 2 and 3



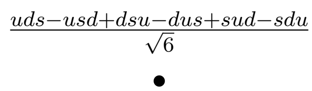
χ_{13} : Antisymmetric in 1 and 3



The Baryon Decuplet

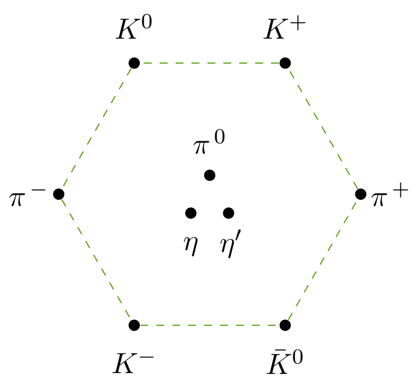


χ_{S} : Completely Symmetric

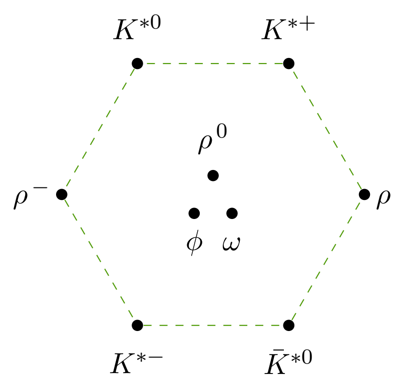


χ_{A} : Completely Antisymmetric

Light Mesons ($\ell = 0$)



Pseudoscalar Meson Nonet
(Octet plus Singlet η')



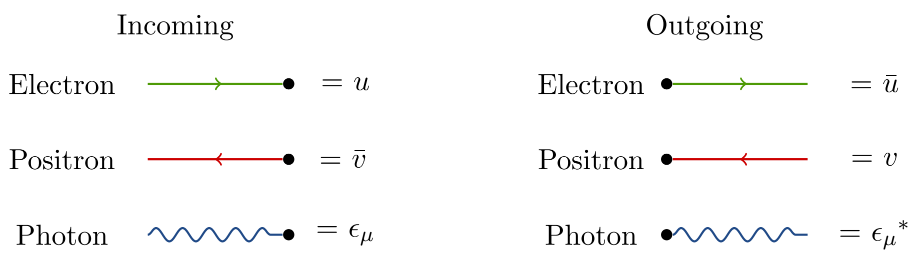
Vector Meson Nonet
(Octet plus Singlet ω)

QED Feynman Rules

1. Make sure that fermion are labelled with an arrow that indicates whether they are particles or anti-particles.

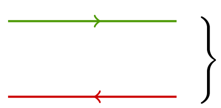
2. Associate a 4-momentum $p_1, p_2, \ldots, p_n$ with each external line, and draw an arrow next to each line pointing forward in time. Associate a 4-momentum $q_1, q_2, \ldots q_m$ with each internal line, and draw an arrow next to each line (the direction is arbitrary for internal line).

3. External lines contribute the following factors



4. Each vertex contributes a factor $i\, g\, \gamma^\mu$, where the dimensionless coupling g is given by $g = e\sqrt{4\pi/\hbar c}$.

5. Internal lines (propagators) contribute the following factors



$$= \frac{i\,(\gamma^\mu q_\mu + m\,c)}{q^2 - m^2\,c^2}$$

$$= \frac{-i\,\eta_{\mu\nu}}{q^2}$$

6. For each vertex, include a factor of the form

$$(2\pi)^4\,\delta^{(4)}(\pm k_1 \pm k_2 \pm k_3) \tag{1}$$

where the k_i are the 4-momenta entering the vertex. The sign of each term is plus for a momentum arrow pointing into the vertex and minus if the momentum arrow points out of the vertex.

7. For each internal line with momentum q include a factor

$$\frac{d^4 q}{(2\pi)^4} \tag{2}$$

and integrate.

8. After performing integrals and using the various delta functions, the result will still contain an overall factor

$$(2\pi)^4\,\delta^{(4)}(p_1 + p_2 + \ldots - p_n)\ . \tag{3}$$

Replace this with a factor of 'i' to obtain the amplitude $\mathcal{M}$.

9. If there are multiple diagrams, include appropriate antisymmetrization factors. Include an extra minus sign between diagrams that differ only by exchange of two incoming (or outgoing) fermions or antifermions, or by exchange of an incoming fermion and outgoing antifermion (or vice versa).

Casimir's Trick

For a process with initial spin states s, final spin states s'

$$\sum_{s,s'} [\bar{u}^{(s)}(p_A)\,\Gamma_1\,u^{(s')}(p_B)]\,[\bar{u}^{(s)}(p_A)\,\Gamma_2\,u^{(s')}(p_B)]^* = \mathrm{Tr}\left[\Gamma_1\,(\not{p}_A + m_A\,c)\,\bar{\Gamma}_2\,(\not{p}_B + m_B\,c)\right]$$
$$\sum_{s,s'} [\bar{v}^{(s)}(p_A)\,\Gamma_1\,v^{(s')}(p_B)]\,[\bar{v}^{(s)}(p_A)\,\Gamma_2\,v^{(s')}(p_B)]^* = \mathrm{Tr}\left[\Gamma_1\,(\not{p}_A - m_A\,c)\,\bar{\Gamma}_2\,(\not{p}_B - m_B\,c)\right]$$
$$\sum_{s,s'} [\bar{u}^{(s)}(p_A)\,\Gamma_1\,v^{(s')}(p_B)]\,[\bar{u}^{(s)}(p_A)\,\Gamma_2\,v^{(s')}(p_B)]^* = \mathrm{Tr}\left[\Gamma_1\,(\not{p}_A + m_A\,c)\,\bar{\Gamma}_2\,(\not{p}_B - m_B\,c)\right]$$
$$\sum_{s,s'} [\bar{v}^{(s)}(p_A)\,\Gamma_1\,u^{(s')}(p_B)]\,[\bar{v}^{(s)}(p_A)\,\Gamma_2\,u^{(s')}(p_B)]^* = \mathrm{Tr}\left[\Gamma_1\,(\not{p}_A - m_A\,c)\,\bar{\Gamma}_2\,(\not{p}_B + m_B\,c)\right]$$

where $u^{(s)}(p)$ and $v^{(s)}(p)$ are Dirac spinors, their conjugates are $\bar{u}^{(s)}(p)$ and $\bar{v}^{(s)}(p)$, and Γ_1 and Γ_2 are any complex 4×4 matrices. Factors of $m\,c$ appearing in the traces on the right-hand side of these relations have implicit factors of the 4×4 identity matrix.

Trace Identities

If A and B are matrices and α is a complex number

$$\mathrm{Tr}\,[\,A + B\,] = \mathrm{Tr}\,[\,A\,] + \mathrm{Tr}\,[\,B\,]$$
$$\mathrm{Tr}\,[\,\alpha\,A\,] = \alpha\,\mathrm{Tr}\,[\,A\,]$$

Cyclic property of the Trace

$$\mathrm{Tr}\,[\,A_1\,A_2\,\ldots A_{n-1}\,A_n\,] = \mathrm{Tr}\,[\,A_n\,A_1\,A_2\,\ldots A_{n-1}\,] = \ldots = \mathrm{Tr}\,[\,A_2\,\ldots A_{n-1}\,A_n\,A_1\,]$$

Gamma Matrix Identities

If a_μ and b_μ are 4-vectors, and $\not{a} = \gamma^\mu\,a_\mu$, then $\{\gamma_\mu, \gamma_\nu\} = 2\,\eta_{\mu\nu}$ implies

$$\not{a}\,\not{b} + \not{b}\,\not{a} = 2\,a \cdot b$$

Other contractions of gamma matrices follow from the anti-commutation relations

$$\gamma_\mu \gamma^\mu = 4$$
$$\gamma_\mu \gamma^\nu \gamma^\mu = -2\,\gamma^\nu$$

The trace of an odd number of gamma matrices is always zero. Some other traces are

$$\mathrm{Tr}\,[\,\mathbb{I}_{4\,\times\,4}\,] = 4$$
$$\mathrm{Tr}\,[\,\gamma^\mu \gamma^\nu\,] = 4\,\eta^{\mu\nu}$$
$$\mathrm{Tr}\,[\,\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma\,] = 4\,\left(\eta^{\mu\nu}\,\eta^{\lambda\sigma} - \eta^{\mu\lambda}\,\eta^{\nu\sigma} + \eta^{\mu\sigma}\,\eta^{\nu\lambda}\right)\;.$$

Factors of the 4×4 identity matrix are implicit in many of these identities.