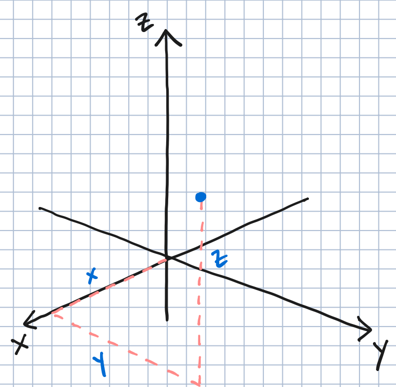
Curvilinear Orthogonal Coordinate Systems

- We will start w/ something you've seen before: Coordinates. You're used to describing where things are in terms of x, y, and z. These are called CARTESIAN COORDS. And you've probably also seen some other examples like SPHERICAL POLAR or CYLINDRICAL POLAR COORDS.

- But before we get into different kinds of coordinates, let's take a few minutes to to review what coordinates are supposed to do, and what they mean.



- COORDINATES are a way of describing a point. ← in space, plane, etc
  To be useful, this description has to be unique and unambiguous.

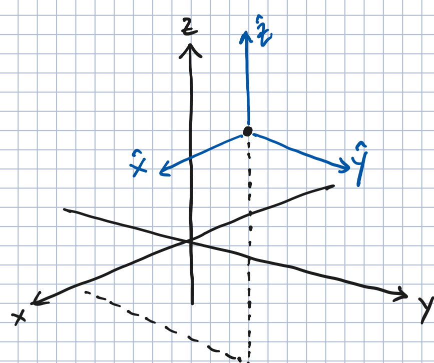
- MANY ways to do this! For instance, two planes intersect along a line, and three planes intersect at a point. So a description like $x = 1m$, $y = 1m$, and $z = 1m$ identifies a pt. by telling you about the intersection of three planes:
  - The y-z plane with $x = 1$
  - The x-z plane $y = 1$
  - The x-y plane $z = 1$

- This probably seems a little complicated for something as simple as x, y, z. The idea is that coordinates give every point an ADDRESS that you know how to interpret.

- So coordinates are a way of assigning a unique set of numbers (the address) to every point. If I tell you about some great new way of describing where things are, but I don't meet these requirements (a unique address for every point) then I don't have a good Coordinate System!

- Once I know how to describe where points are, I can do things like tell you about the location of something (a baseball, say) at different times. I'd do that by giving you 3 functions of t - one for each number in the address. If I were using Cartesian coords this would be $x(t)$, $y(t)$, and $z(t)$.

- But Cartesian coords aren't always the most useful! For example, if I was describing the motion of a satellite around the Earth, I might want to use its altitude, latitude, and longitude. Why? Because I expect some of those quantities might stay constant throughout its orbit, while all 3 of x, y, z would change.
- Depending on the problem, some coordinates are better-suited to what you are trying to do!
- Now, in both those examples, each coord. is independent of the other two. That is, moving in the x-dir doesn't change the y or z coord. Likewise, you can imagine increasing an object's altitude without changing its latitude or longitude. At any point there are 3 directions you can go in, and they are all "perpindicular."
- Mathematically, we state this by writing down 3 unit vectors for each direction. We use the dot product to check whether they are $\perp$. In Cart. Coords we'll call them $\hat{x}, \hat{y}, \hat{z}$:



$$\hat{x}\cdot\hat{x}=1 \quad \hat{x}\cdot\hat{y}=0 \quad \hat{x}\cdot\hat{z}=0$$
$$\hat{y}\cdot\hat{x}=0 \quad \hat{y}\cdot\hat{y}=1 \quad \hat{y}\cdot\hat{z}=0$$
$$\hat{z}\cdot\hat{x}=0 \quad \hat{z}\cdot\hat{y}=0 \quad \hat{z}\cdot\hat{z}=1$$

Remember: Unit vectors have magnitude (length) = 1!

- In other coord systems they'll have diff. names, but there'll always be 3 of them (or 2 in a plane) b/c you need to describe 3 possible directions.
- So imagine I describe some coord. system to you, and let's call the 3 unit vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$. If all 3 are perpindicular:

$$\hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

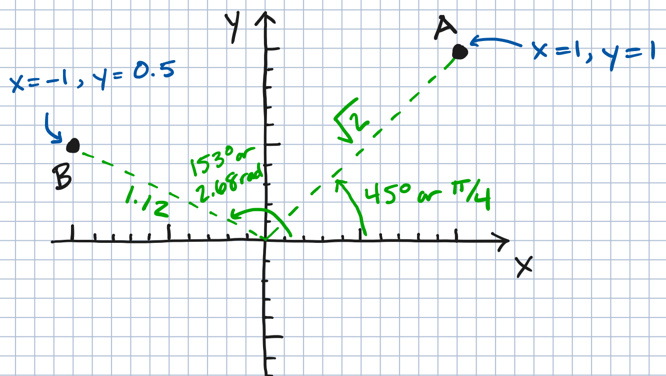
Then we have an ORTHOGONAL COORDINATE SYSTEM. (Orthogonal is just another word for perpindicular.)

CARTESIAN, SPC, and CYLINDRICAL POLAR COORDS are all OCS.

- As I said before, some coord. systems are more useful than others, depending on the problem or application. So knowing how to translate between different coord. systems is essential.
- Most of the coords we use in physics are OCS, so we're going to learn how to translate statements about vectors & coords from one OCS into another!

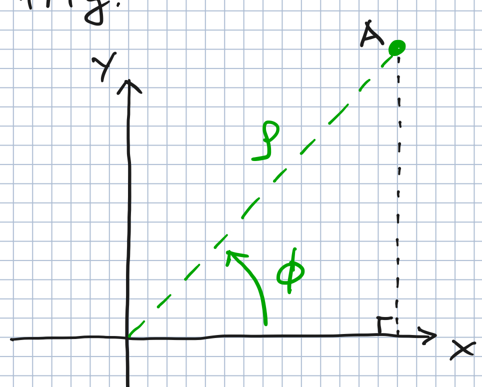
DISTANCES, SCALE FACTORS, AND DISPLACEMENTS

- Let's start simple, working in just a plane. You're used to x-y coords. But you've also seen POLAR coords:



- Pt. A is located @ $x=1$, $y=1$. Or, you could also say it is $\sqrt{2}$ from the origin, @ an angle of $\pi/4$ (CCW) from the x-axis.
- Pt. B is 1.12 from the origin, and the angle is 2.68 (radians!)

- We can describe every pt. in the plane by specifying its distance from the origin, and an angle CCW from the x-axis. (The choice of x-axis for the angle is arbitrary. We could use any other line, but we will always use the x-axis so there's no ambiguity!)
- And we can relate the two descriptions of a point using a little trig:



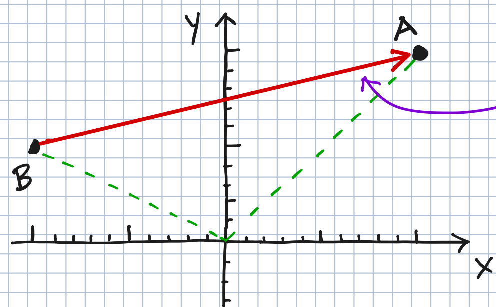
$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$

We'll use ρ for distance from the origin, and ϕ for the angle.

IMPORTANT: x, y and ρ, ϕ are two equivalent descriptions of point A.

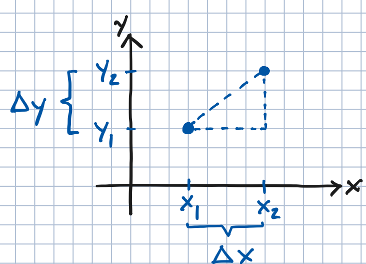
- Now, when we change from one coord. system to another, the way we describe a particular point will change.

- But the points themselves don't change! So things like the distance b/t two points, or the arrow you draw to point from one to the other, those things <u>do not</u> change. We're going to make use of this to help us figure out how to translate various quantities b/t two OCS.



You draw the same arrow b/t A & B, and it has the same length, whether you use Cartesian or Polar coords.

- We'll start by considering two pts in the plane that are very close to each other. One has Cart. coords (x_1, y_1) & the other is (x_2, y_2).



- How far apart are they? The Pythagorean thm. tells you that:

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

- Now imagine that they are <u>very</u> close together – so close that Δx & Δy essentially become the infinitesimal dx & dy you're familiar with from Calc. Then:

$$ds^2 = dx^2 + dy^2$$

- What if we want to express this in <u>polar coords</u>? The dx b/t the two pts could be due to diff. values of ρ, or of ϕ, or both:

$$x = \rho\cos\phi \Rightarrow \underbrace{dx = \frac{\partial x}{\partial \rho}d\rho + \frac{\partial x}{\partial \phi}d\phi}_{\substack{\text{Total der. of a function } (x) \\ \text{of two variables } (\rho \,\&\, \phi) \\ \text{from multi-variable calc.}}} \Rightarrow dx = \cos\phi\, d\rho - \rho\sin\phi\, d\phi$$

- Likewise for dy:

$$y = \rho\sin\phi \Rightarrow dy = \frac{\partial y}{\partial \rho}d\rho + \frac{\partial y}{\partial \phi}d\phi = \sin\phi\, d\rho + \rho\cos\phi\, d\phi$$

- Since we know how dx & dy relate to $d\rho$ & $d\phi$, & we know how to express the distance b/t the points in terms of dx & dy:

$$ds^2 = (\cos\phi\, d\rho - \rho\sin\phi\, d\phi)^2 + (\sin\phi\, d\rho + \rho\cos\phi\, d\phi)^2$$

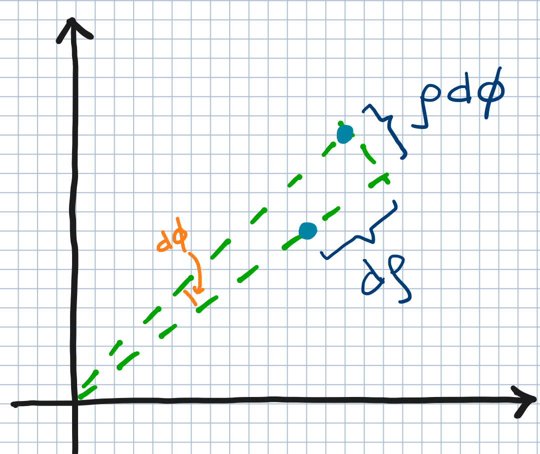
$$= (\underbrace{\cos^2\phi + \sin^2\phi}_{1})\, d\rho^2 + (\cancel{-2\cos\phi\sin\phi + 2\cos\phi\sin\phi})\, d\rho\, d\phi$$

$$+ (\rho^2\sin^2\phi + \rho^2\cos^2\phi)\, d\phi^2$$

$$ds^2 = d\rho^2 + \rho^2 d\phi^2$$

- So, now we know how to translate our expression for the distance b/t two infinitesimally separated points into polar coords:

$$dx^2 + dy^2 = d\rho^2 + \rho^2 d\phi^2$$



Two things to notice:

1) No $d\rho\, d\phi$ term in ds^2.

2) It's '$\rho\, d\phi$' that shows up, not $d\phi$ by itself. That's b/c $d\phi$ is just a change in the polar angle. The <u>distance</u> is $\rho d\phi$!

- Now suppose I tell you about 2 new coords that I'll call q_1 & q_2. If this is an OCS then ds^2 will have a dq_1^2 term and a dq_2^2 term, but <u>no</u> $dq_1\, dq_2$ term.

- Here's an example: Coords u, v related to x & y by

$$x = \frac{1}{2}(u^2 - v^2) \qquad y = uv$$ ← 'PARABOLIC COORDINATES'

↳ $dx = u\,du - v\,dv \qquad dy = du\, v + u\, dv$

↳ $$dx^2 + dy^2 = (u\,du - v\,dv)^2 + (v\,du + u\,dv)^2$$

$$= (u^2 + v^2)\,du^2 + (\cancel{-2uv + 2uv})\,du\,dv$$

$$+ (u^2 + v^2)\,dv^2$$

$$\Rightarrow ds^2 = (u^2 + v^2)(du^2 + dv^2)$$

- Like Cartesian & Polar, these coords have du^2 & dv^2 showing up in ds^2, but <u>not</u> $du\,dv$.
- In other words, u & v are perpendicular, just like x & y or ρ & ϕ.
- But notice that the <u>coefficients</u> of du^2 & dv^2 look <u>very</u> different than the coeff of dx^2 & dy^2, or $d\rho^2$ & $d\phi^2$.
- We call the $\sqrt{\ldots}$ of these coefficients 'SCALE FACTORS.'
- So if our coords are q_1 & q_2, & the scale factors are h_1 & h_2, then

$$\boxed{ds^2 = h_1(q_1,q_2)^2\,dq_1^{\,2} + h_2(q_1,q_2)^2\,dq_2^{\,2}}$$

Note that the scale factors are <u>squared</u> in ds^2! See below.

The scale factors can be constants, or functions of one of the coords, or functions of both!

- For our previous examples:

CARTESIAN: $\begin{cases} h_x = 1 \\ h_y = 1 \end{cases}$

POLAR: $\begin{cases} h_\rho = 1 \\ h_\phi = \rho \end{cases}$

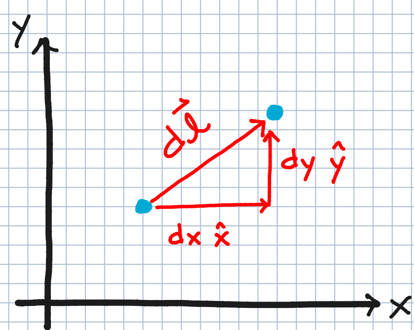
PARABOLIC: $\begin{cases} h_u = \sqrt{u^2+v^2} \\ h_v = \sqrt{u^2+v^2} \end{cases}$

Sometimes we use numerals 1, 2 to denote which coord; sometimes we use the name of the coord ('x' or 'ρ', etc).

- The POLAR coord example already shows us what the scale factors <u>mean</u>. If we move b/t two pts separated by angle $d\phi$ in the ϕ-direction, the corresponding distance is $\rho\, d\phi$, not $d\phi$.
- Likewise, in PARABOLIC coords, if two pts have the same v coord. but their u coords differ by du, the distance b/t them is $du\sqrt{u^2+v^2}$.

- So the scale factors show us how actual <u>distances</u> are related to the way the coordinates change when we move in various lirections.

- What can we do with this? Well, for starters, we can say something about the distances associated w/ moving in various directions, so we can write out the <u>displacement</u> vector b/t two infinitesimally nearby points:



our notation

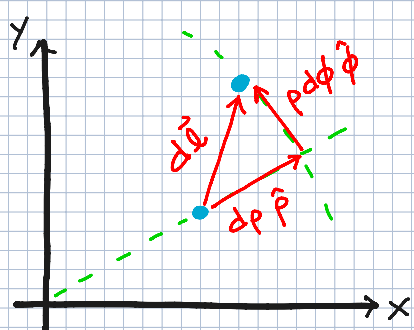
Dist. moved in y-dir...

$$d\vec{l} = dx\ \hat{x} + dy\ \hat{y}$$

... unit vector that indicates y-dir.

Dist. moved in x-dir...

... unit vector that indicates x-dir.



ρ-dir.

ϕ-dir.

$$d\vec{l} = d\rho\ \hat{\rho} + \underbrace{\rho\, d\phi}\ \hat{\phi}$$

How far you move due to a small change in ρ

How far you move due to a small change in ϕ, when dist. from origin is ρ

- More generally, if you have an OCS q_1, q_2 w/ scale factors h_1 & h_2, then $d\vec{l}$ is:

$$d\vec{l} = \underbrace{h_1\, dq_1}\ \hat{q}_1 + h_2\, dq_2\ \hat{q}_2$$

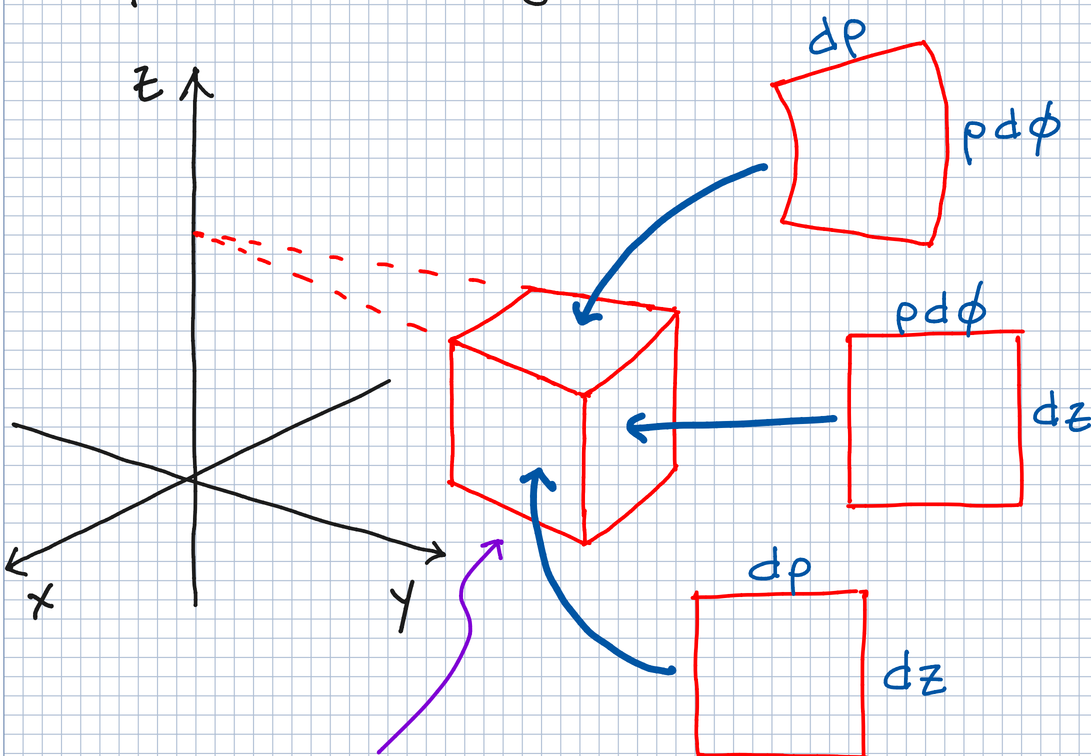
Remember, the scale factors can be constants or functions. It depends on the coord. system.

Distance b/t two pts. sep. by dq_1 in the $\hat{q}_1$ direction.

- Wait, we're familiar w/ $\hat{x}$ & $\hat{y}$, but what about $\hat{\rho}$ & $\hat{\phi}$? And what do $\hat{q}_1$ & $\hat{q}_2$ mean?

- Short answer: $\hat{q}_1$ is the direction you go in if you increase q_1. So $\hat{\rho}$ is radially outward; $\hat{\phi}$ is CCW. We'll discuss this in more detail in a moment!

- Besides helping us understand how to express the infinitesimal displacement $d\vec{l}$ in different OCS, the scale factors also help us sort out area and volume elements for integrals.
- Consider CYLINDRICAL POLAR COORDINATES. They're just polar coords, along w/ z:



When z is constant, little patches of area have

$$dA = \rho\, d\phi\, d\rho$$

... when ρ is constant, it's

$$dA = \rho\, d\phi\, dz$$

... and when ϕ is constant it's

$$dA = d\rho\, dz$$

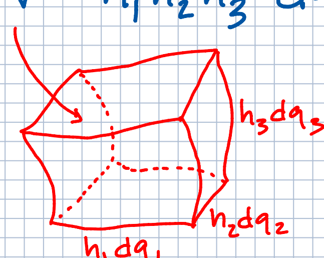
The volume element in CPC is:

$$dV = \rho\, d\phi\, d\rho\, dz$$

- The dA's are little rectangles, and the dV is a little rectangular prism. The scale factor 'ρ' shows up in the length of the side associated w/ changing ϕ.
- If we have an OCS q_1, q_2, q_3 w/ scale factors $h_1, h_2,$ and h_3, then the area & volume elements are:

$$dA = \begin{cases} h_1 h_2\, dq_1\, dq_2 & (\text{const. } q_3) \\ h_1 h_3\, dq_1\, dq_3 & (\text{const. } q_2) \\ h_2 h_3\, dq_2\, dq_3 & (\text{const. } q_1) \end{cases}$$

$$dV = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$$



- This might look complicated, but think about what we're doing: it's just some statements about three possible lengths, one for each of the three ways you could change one of the coordinates.

$$d\ell_1 = h_1\, dq_1$$
$$d\ell_2 = h_2\, dq_2$$
$$d\ell_3 = h_3\, dq_3$$

If you're on a surface where q_1 is constant (like $\rho =$ const. on the wall of a cylinder, or $r =$ const. on surface of a sphere) then the two remaining ways you can move have lengths $h_2\, dq_2$ & $h_3\, dq_3$; those are the sides of an infinitesimal rectangle w/ area $dA = h_2 h_3\, dq_2\, dq_3$, etc.

OCS And Basis Vectors

- When we were talking about $d\vec{\ell}$, we said something like '$\hat{q}_1$ is the direction you go in when you increase q_1.' So $\hat{x}$ is the dir. you move in when you increase x, $\hat{\rho}$ is moving outward from the origin in the plane (or the z-axis in 3-D), etc.

- But is that useful? What if I told you about a vector $\vec{A}$ by describing it's components in Cartesian coords:

$$\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}$$

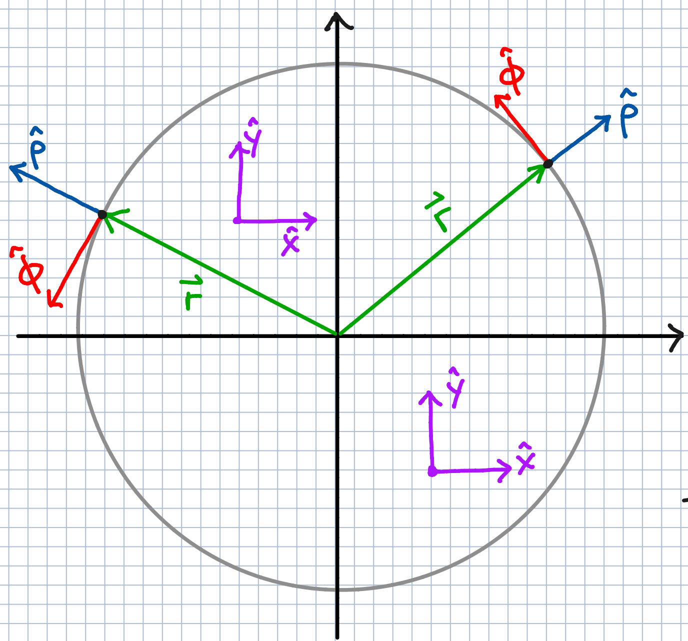
- Could you then give me a description of the vector in some other OCS?

$$A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z} = \overbrace{A_\rho\,\hat{\rho} + A_\phi\,\hat{\phi}}^{?} + A_z\,\hat{z}$$

- No; we need some way of converting Cart. basis vectors into CPC (or other OCS) basis vectors. How do we do that?

- First, let's try to visualize $\hat{\rho}$ & $\hat{\phi}$.

- We know that $\hat{\rho}$ moves radially outward from the origin, & $\hat{\phi}$ moves CCW.



- Notice that $\hat{x}$ & $\hat{y}$ point in the same dir <u>everywhere</u>. That is, $\hat{x}$ looks the same @ every pt, same for $\hat{y}$.

- Not so for $\hat{\rho}$ & $\hat{\phi}$! They <u>mean</u> the same thing everywhere, but they <u>look</u> different.

- Note, though, that they are still <u>unit vectors</u>, and they are still $\perp$ to each other:

$$\hat{\rho}\cdot\hat{\rho} = 1 \qquad \hat{\phi}\cdot\hat{\phi} = 1 \qquad \hat{\rho}\cdot\hat{\phi} = 0$$

Note that I'm talking about $\hat{\rho}$ & $\hat{\phi}$ @ a specific pt. I'm <u>not</u> comparing $\hat{\rho}$ at one pt. to $\hat{\rho}$ @ another pt, etc.

- Okay, so it looks like $\hat{x}$ & $\hat{y}$ are constant, while $\hat{\rho}$ & $\hat{\phi}$ 'look' different @ different points. How can I relate them? That is, how do I <u>convert</u> from $\hat{x}, \hat{y}$ to $\hat{\rho}, \hat{\phi}$ & vice-versa?

- Consider a pt. w/ Cart. coords x, y. The <u>position vector</u> for that point is

$$\vec{r} = x\,\hat{x} + y\,\hat{y}$$

- And we know how to write x & y in terms of ρ & ϕ:

$$\vec{r} = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y}$$

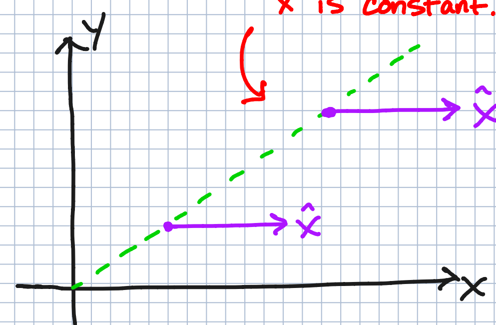
- Now, if we change ρ a little bit, we know that moves us to a nearby pt. separated from the 1st point by $d\rho\,\hat{\rho}$, right? So let's look @ the <u>derivative</u> of $\vec{r}$ with respect to ρ.

$$\vec{r} = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} \rightarrow \frac{d\vec{r}}{d\rho} = \frac{d}{d\rho}(\rho\cos\phi\,\hat{x}) + \frac{d}{d\rho}(\rho\sin\phi\,\hat{y})$$

$$\frac{d}{d\rho}(\rho\cos\phi\,\hat{x}) = \cos\phi\,\hat{x} + \rho\underbrace{\frac{d}{d\rho}(\cos\phi)}_{=0,\ b/c\ d\phi/d\rho=0}\hat{x} + \rho\cos\phi\underbrace{\frac{d\hat{x}}{d\rho}}_{=0\ b/c\ \hat{x}\text{ is constant.}}$$

$$\hookrightarrow \frac{d}{d\rho}(\rho\cos\phi\,\hat{x}) = \cos\phi\,\hat{x}$$

$$\frac{d}{d\rho}(\rho\sin\phi\,\hat{y}) = \sin\phi\,\hat{y}$$



- So if we change ρ by $d\rho$, the position vector changes by:

$$d\vec{r} = (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})\,d\rho$$

- The part in front of $d\rho$ must be what we mean by $\hat{\rho}$!

$$\hookrightarrow \hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

Check: $\hat{\rho}\cdot\hat{\rho} = 1$, as we expect for a unit vector?

$$\hat{\rho}\cdot\hat{\rho} = \cos^2\phi\,\cancelto{1}{\hat{x}\cdot\hat{x}} + 2\cos\phi\sin\phi\,\cancelto{0}{\hat{x}\cdot\hat{y}} + \sin^2\phi\,\cancelto{1}{\hat{y}\cdot\hat{y}}$$

$$= \cos^2\phi + \sin^2\phi = 1\ \checkmark$$

- Let's try this w/ ϕ & $\hat{\phi}$!

$$\frac{d\vec{r}}{d\phi} = -\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y}$$

As before, $\frac{d\rho}{d\phi} = 0$, and $\frac{d\hat{x}}{d\phi} = \frac{d\hat{y}}{d\phi} = 0$.

$$\hookrightarrow d\vec{r} = \rho(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})\,d\phi$$

- Is $\hat{\phi}$ the part in front of $d\phi$? Not quite; check it's length!

$$(-\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y})\cdot(-\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y}) = \rho^2(\sin^2\phi + \cos^2\phi) = \rho^2$$

- So to get $\hat{\phi}$ we need to divide the stuff in front of $d\phi$ by its magnitude $\sqrt{\rho^2} = \rho$:

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

We divided by $\left|\frac{d\vec{r}}{d\phi}\right|$, which is the scale factor $h_\phi = \rho$. Is it clear why?

- And now we know how to convert b/t Cart. directions & polar directions!

$$\hat{\rho} = \frac{1}{\left|\frac{d\vec{r}}{d\rho}\right|}\frac{d\vec{r}}{d\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = \frac{1}{\left|\frac{d\vec{r}}{d\phi}\right|}\frac{d\vec{r}}{d\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

Can you invert this & write $\hat{x}$ & $\hat{y}$ as combinations of $\hat{\rho}$ & $\hat{\phi}$?

- Now, we said that POLAR coords are an OCS, and that was supposed to mean that the unit vectors are perpindicular. Is that the case?

$$\hat{\rho}\cdot\hat{\phi} = (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})\cdot(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0 \checkmark$$

- For any other OCS, we'd do the same thing. First, write x & y in terms of q_1 & q_2, & then:

$$\hat{e}_i = \frac{1}{\left|\frac{d\vec{r}}{dq_i}\right|}\frac{d\vec{r}}{dq_i} = \frac{1}{h_i}\frac{d\vec{r}}{dq_i}$$

Remember, i is just a label (or 'index') that we use to indicate which coord. we're talking about.

Dividing by the magnitude insures that we get a <u>unit</u> vector, as we saw in the $\hat{\phi}$ example.

- Let's do one more detailed example to make sure this is all clear. We'll work out the unit vectors for the PARABOLIC coords from earlier.

- We defined PARABOLIC coords as

$$x = \frac{1}{2}(u^2 - v^2) \qquad y = uv$$

This replaces the usual Cartesian grid w/ a 'grid' made out of a bunch of parabolas. Parabolic coords show up sometimes in <u>orbital</u> <u>mechanics</u>.

- Before we work out the unit vectors, let's take a moment to think about what these coordinates look like. What sort of grid do they make?

- First, you need to know that we always assume that one of the parabolic coords is <u>non-negative</u>. We'll always use non-negative u: $0 \le u < \infty$. Then $-\infty < v < \infty$.

- (Why? Otherwise pairs like $u=4, v=3$ and $u=-4, v=-3$ would refer to the same x, y point! Coords should give every pt. a <u>unique</u> address. Here we deal w/ that by assuming u is never negative. That's not the only way to do it, but there's no need to dive into that right now.)

- So how do we visualize Parabolic coords? We think of Cartesian coords as a grid b/c $x=3$ is a vertical line, and $y=-2$ is a horizontal line, etc. So what do we get when we set $u=3$ or $v=-2$?

- Well, what pts in the plane correspond to a constant value of v?

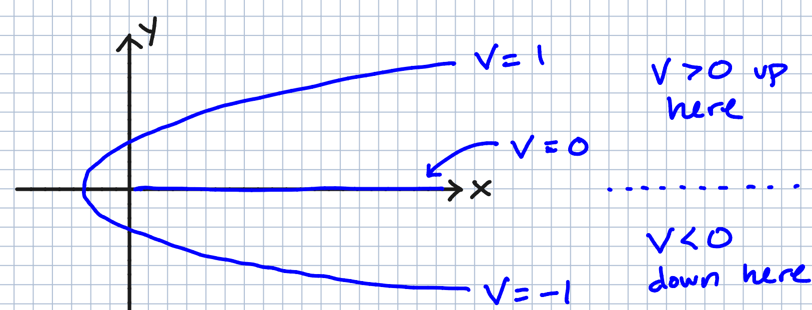
Use '+' b/c we decided $u \geq 0$!

$$x = \frac{1}{2}u^2 - \frac{1}{2}v^2 \rightarrow u^2 = v^2 + 2x \Rightarrow u = \pm\sqrt{v^2+2x}$$

$$y = uv = v\sqrt{v^2+2x}$$

$$\hookrightarrow y = v\sqrt{v^2+2x}$$

The top ($v>0$) or bottom ($v<0$) half of a horizontal parabola.



- Likewise, what points in the plane correspond to a constant value of u?

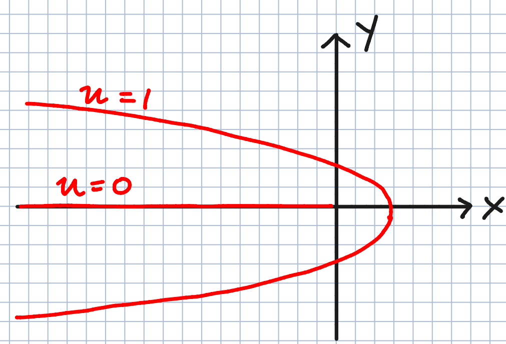
$$x = \frac{1}{2}u^2 - \frac{1}{2}v^2 \rightarrow v^2 = u^2 - 2x \Rightarrow v = \pm\sqrt{u^2-2x}$$

v can be either pos. or neg.

$$y = uv = \pm u\sqrt{u^2-2x}$$

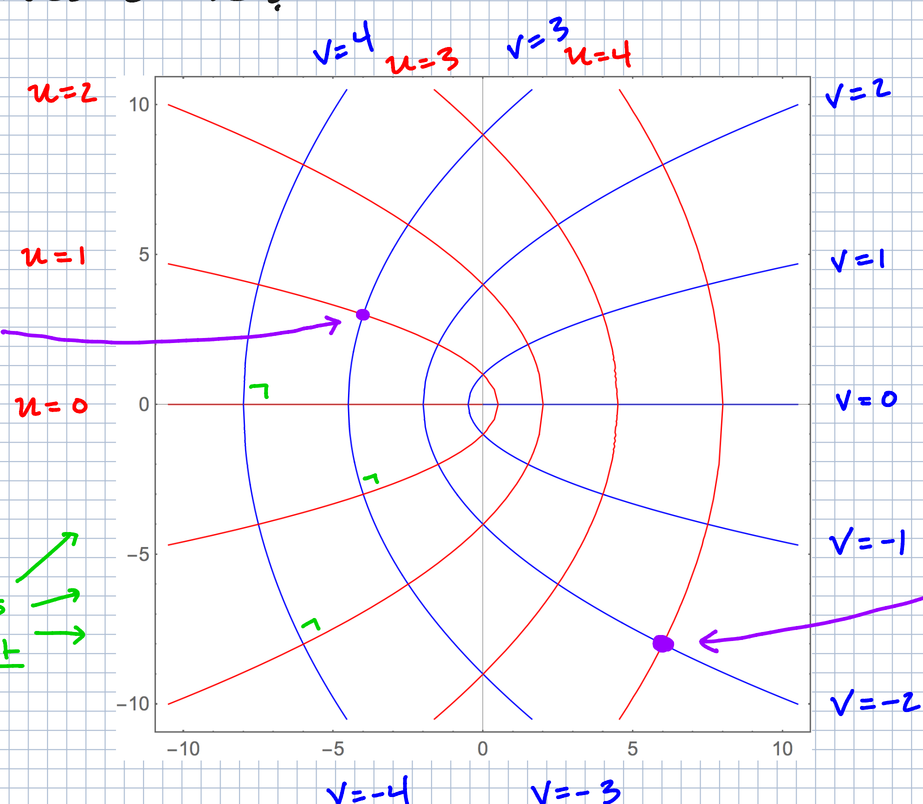
$$\hookrightarrow \underbrace{y = \pm u\sqrt{u^2-2x}}$$

$u \geq 0$, but since we have $\pm$ we get both halves of the parabola.



- So now we can visualize the Parabolic coordinates grid by plotting several of these parabolas corresponding to diff. values of u & v.

$u=1$, $v=3$ gives $x=-4$, $y=3$



$u=4$, $v=-2$ gives $x=6$, $y=-8$

Notice that the parabolas always intersect @ right angles.

- Back to our calculation! We already worked out the scale factors:

$$\left.\begin{aligned} dx &= u\,du - v\,dv \\ dy &= v\,du + u\,dv \end{aligned}\right\} \Rightarrow ds^2 = (u^2+v^2)(du^2+dv^2)$$

$$\Rightarrow h_u = h_v = \sqrt{u^2+v^2}$$

– Now let's write out the position vector, w/ x & y in terms of u & v, and find the unit vectors.

$$\vec{r} = x\hat{x} + y\hat{y} = \frac{1}{2}(u^2 - v^2)\hat{x} + uv\,\hat{y}$$

$$\frac{d\vec{r}}{du} = \underbrace{u}_{\frac{dx}{du}}\hat{x} + \underbrace{v}_{\frac{dy}{du}}\hat{y} \qquad \left|\frac{d\vec{r}}{du}\right| = \sqrt{u^2+v^2} \stackrel{\checkmark}{=} h_u$$

I know I'm supposed to get h_u & h_v here; this is a nice check of my work.

$$\Rightarrow \hat{u} = \frac{u}{\sqrt{u^2+v^2}}\hat{x} + \frac{v}{\sqrt{u^2+v^2}}\hat{y}$$

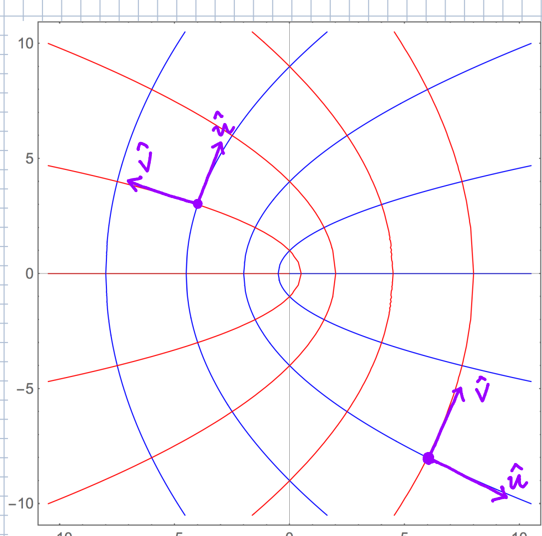
$$\frac{d\vec{r}}{dv} = \overbrace{-v}^{\frac{dx}{dv}}\hat{x} + \overbrace{u}^{\frac{dy}{dv}}\hat{y} \qquad \left|\frac{d\vec{r}}{dv}\right| = \sqrt{u^2+v^2} \stackrel{\checkmark}{=} h_v$$

$$\Rightarrow \hat{v} = -\frac{v}{\sqrt{u^2+v^2}}\hat{x} + \frac{u}{\sqrt{u^2+v^2}}\hat{y}$$

$$\hat{u}\cdot\hat{v} = -\frac{uv}{u^2+v^2} + \frac{vu}{u^2+v^2} \stackrel{\checkmark}{=} 0$$

As a final check, I confirm that $\hat{u}$ & $\hat{v}$ are orthogonal.

– So PARABOLIC coords are indeed an OCS. We can draw $\hat{u}$ & $\hat{v}$ @ a few pts.



$u=1$, $v=3$:
$$\hat{u} = \frac{1}{\sqrt{10}}\hat{x} + \frac{3}{\sqrt{10}}\hat{y}, \quad \hat{v} = -\frac{3}{\sqrt{10}}\hat{x} + \frac{1}{\sqrt{10}}\hat{y}$$

$u=4$, $v=-2$
$$\hat{u} = \frac{2}{\sqrt{5}}\hat{x} - \frac{1}{\sqrt{5}}\hat{y}, \quad \hat{v} = \frac{1}{\sqrt{5}}\hat{x} + \frac{2}{\sqrt{5}}\hat{y}$$

← $\hat{u}$ always pts in the direction of increasing u, and $\hat{v}$ in the dir. of increasing $\hat{v}$. Notice that, even though the x & y components of $\hat{u}$ & $\hat{v}$ change, they always have the same <u>relative</u> orientation.

- As a check, what if I proposed some new coords related to Cartesian coords by

$$x = \frac{1}{2}(\alpha^2 + \beta^2) \qquad y = \alpha\beta$$

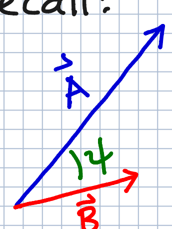
- They look pretty similar to PARABOLIC coords. But are they an OCS? Do you need to derive the unit vectors to check? No. Give me one simple reason why this is not an OCS. (HINT: Look @ ds^2.) Now, give me an even simpler reason why it's not even a good coord. system!

TRANSLATING A VECTOR FROM CARTESIAN TO AN OCS

- We started the last section by asking how, given the x, y, z components of a vector, we would go about describing it in some other OCS:

$$\vec{A} = \underbrace{A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}}_{\text{Given these...}} = \underbrace{A_1\,\hat{e}_1 + A_2\,\hat{e}_2 + A_3\,\hat{e}_3}_{\text{What are } A_1, A_2, A_3?}$$

- And now we're equipped to answer it!
- First, how do we determine the components of a vector? The A_1 component of $\vec{A}$ is 'how much' of $\vec{A}$ is in the $\hat{e}_1$ direction, and likewise for A_2 & A_3. We determine that with the dot product of $\vec{A}$ & each unit vector.
- Recall:



$$\leftarrow \underbrace{\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\psi}_{\substack{\text{So if } \vec{B} \text{ is a unit vector,}\\ \text{this gives the fraction of}\\ |\vec{A}| \text{ in the dir. along } \vec{B}.}}$$

Angle b/t $\vec{A}$ & $\vec{B}$

$$\Rightarrow \quad A_1 = \vec{A}\cdot\hat{e}_1 \qquad A_2 = \vec{A}\cdot\hat{e}_2 \qquad A_3 = \vec{A}\cdot\hat{e}_3$$

- Once we know how to express the $\hat{e}_i$ in terms of $\hat{x}, \hat{y}, \hat{z}$, we can evaluate those dot products.

- Let's look @ our POLAR coords example. There, we found:

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \qquad \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

- So if I gave you the Cart. components A_x & A_y of some vector, then:

$$A_\rho = \hat{\rho}\cdot\vec{A} = (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})\cdot(A_x\,\hat{x} + A_y\,\hat{y})$$
$$= A_x\cos\phi + A_y\sin\phi$$

$$A_\phi = \hat{\phi}\cdot\vec{A} = (-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})\cdot(A_x\,\hat{x} + A_y\,\hat{y})$$
$$= -A_x\sin\phi + A_y\cos\phi$$

$$\Rightarrow \vec{A} = (A_x\cos\phi + A_y\sin\phi)\,\hat{\rho} + (-A_x\sin\phi + A_y\cos\phi)\,\hat{\phi}$$

The Cart. comp. A_x & A_y may just be numbers, or they could be some functions of x & y, in which case you might want to re-write them using $x = \rho\cos\phi$ & $y = \rho\sin\phi$!

- As an example, consider the vector $\vec{A} = 4\,\hat{x} + 7\,\hat{y}$:

$$\hookrightarrow \vec{A} = (4\cos\phi + 7\sin\phi)\,\hat{\rho} + (-4\sin\phi + 7\cos\phi)\,\hat{\phi}$$

Notice that the A_ρ & A_ϕ components change when ϕ changes. What? Isn't it a constant vector? YES. Remember that $\hat{\rho}$ & $\hat{\phi}$ change, too!

- A more <u>interesting</u> example is the POSITION VECTOR. How do we write $\vec{r}$ in polar coords?

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Let's do 3-D CYLINDRICAL POLAR COORDS.

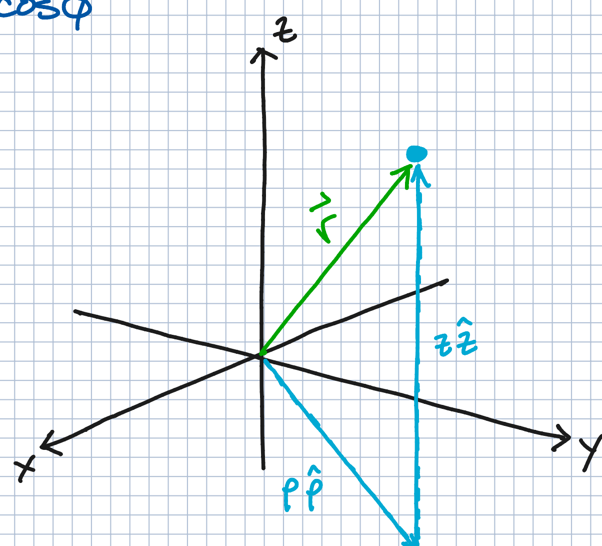
$$r_\rho = \hat{\rho}\cdot\vec{r} = (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})\cdot(x\hat{x} + y\hat{y} + z\hat{z})$$
$$= x\cos\phi + y\sin\phi$$
$$= \rho\cos^2\phi + \rho\sin^2\phi$$
$$= \rho$$

$$\left.\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{aligned}\right\}$$

$$r_\phi = \hat{\phi}\cdot\vec{r} = (-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})\cdot(x\hat{x} + y\hat{y} + z\hat{z})$$
$$= -x\sin\phi + y\cos\phi$$
$$= -\rho\cos\phi\sin\phi + \rho\sin\phi\cos\phi$$
$$= 0$$

$$r_z = \hat{z}\cdot\vec{r} = z$$

$$\Rightarrow \vec{r} = \rho\hat{\rho} + z\hat{z}$$



<u>IMPORTANT</u>: $\vec{r}$ does <u>not</u> have a $\hat{\phi}$ component. Why is this?

- We can work out the components of any vector in any OCS this way; we just need to work out the relationships b/t the basis vectors in the two coord. systems.

Velocity, Acceleration, And Keeping Track Of Changing Unit Vectors

- One of the nice things about Cartesian coords is that $\hat{x}, \hat{y}, \&\ \hat{z}$ are <u>constant</u>; that is, @ any pts (x_1, y_1, z_1) & (x_2, y_2, z_2) they look exactly the same.

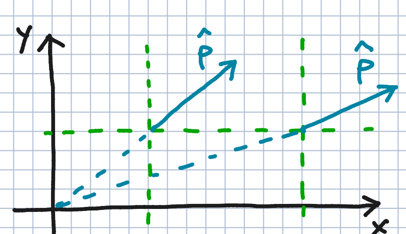
- Another way of stating this idea, that the Cart. basis vectors don't change from pt. to pt., is to say that their derivatives are zero (like any constant!)

$$\frac{d\hat{x}}{dx} = \frac{d\hat{x}}{dy} = \frac{d\hat{x}}{dz} = 0 \quad , \ \& \text{ similar for } \hat{y} \ \& \ \hat{z}$$

- But this isn't true for the other OCS we've seen; vectors like $\hat{\rho}$ & $\hat{\phi}$ seem to change direction (but <u>not</u> length – they're always unit vectors) from pt. to pt.

- (NOTE: There's a bit of a subtle point here. A unit vector like $\hat{\rho}$ or $\hat{\phi}$ <u>means</u> the same thing @ every pt; $\hat{\rho}$ means 'radially away from the origin no matter where you are. So in that sense all OCS basis vectors are 'constant.' Here, when we say a vector changes from pt. to pt. we mean that the arrows you draw @ each pt. look different. But you don't need to worry about this distinction in this class!)

- As an example, consider the vector $\hat{\rho}$ @ two pts. w/ the same y-coord. but diff. x-coords:



Both unit vectors, but you can <u>see</u> that the directions (orientations of arrows) are different.

If a quantity changes as we change x, that means:

$$\frac{d\hat{\rho}}{dx} \neq 0$$

- Well, we know how to write $\hat{\rho}$ in terms of the constant unit vectors $\hat{x}$ & $\hat{y}$, so let's check this:

$$\hat{\rho} = \cos\phi\, \hat{x} + \sin\phi\, \hat{y} = \frac{x}{\sqrt{x^2+y^2}}\,\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\,\hat{y}$$

$$\frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{dy}{dx} = 0$$

$$\hookrightarrow \frac{d\hat{\rho}}{dx} = \left(\frac{1}{\rho} - \frac{x}{\rho^2}\frac{d\rho}{dx}\right)\hat{x} + \left(\frac{0}{\rho} - \frac{y}{\rho^2}\frac{d\rho}{dx}\right)\hat{y}$$

$$\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x = \frac{x}{\rho}$$

$$= \left(\frac{1}{\rho} - \frac{x^2}{\rho^3}\right)\hat{x} + \left(-\frac{xy}{\rho^3}\right)\hat{y}$$

$$= \left(\frac{1}{\rho} - \frac{\rho^2\cos^2\phi}{\rho^3}\right)\hat{x} - \frac{\rho^2\cos\phi\sin\phi}{\rho^3}\,\hat{y}$$

$$= \frac{\sin^2\phi}{\rho}\,\hat{x} - \frac{\cos\phi\sin\phi}{\rho}\,\hat{y} = \frac{\sin\phi}{\rho}\left(\sin\phi\,\hat{x} - \cos\phi\,\hat{y}\right)$$

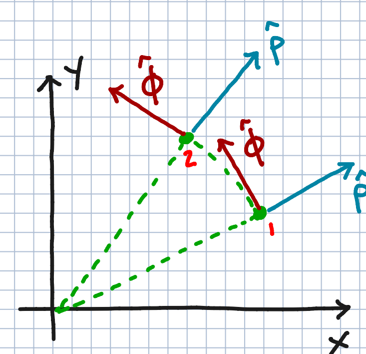
$$\Rightarrow \frac{d\hat{\rho}}{dx} = \frac{\sin\phi}{\rho}\left(\sin\phi\,\hat{x} - \cos\phi\,\hat{y}\right)$$

Makes sense! On last page saw that for $0 \le \phi < \pi/2$, increasing x made $\hat{\rho}$ longer in x-dir. & shorter in y-dir.

- Another example is how $\hat{\rho}$ & $\hat{\phi}$ change as we move CCW around the origin:

$$\frac{d\hat{\rho}}{d\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} = \hat{\phi}$$

$$\frac{d\hat{\phi}}{d\phi} = -\cos\phi\,\hat{x} - \sin\phi\,\hat{y} = -\hat{\rho}$$

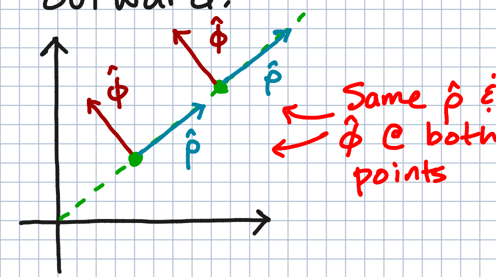


Do these $\approx$ describe change in $\hat{\rho}$ & $\hat{\phi}$ b/t pts. 1 & 2?

- Note, however, that $\hat{\rho}$ & $\hat{\phi}$ <u>don't</u> change when we move in the $\hat{\rho}$ direction – radially inward or outward:

$$\frac{d\hat{\rho}}{d\rho} = \frac{d}{d\rho}(\cos\phi\,\hat{x}) + \frac{d}{d\rho}(\sin\phi\,\hat{y}) = 0$$

$$\frac{d\hat{\phi}}{d\rho} = \frac{d}{d\rho}(-\sin\phi\,\hat{x}) + \frac{d}{d\rho}(\cos\phi\,\hat{y}) = 0$$



- Why do we care about this? First, b/c these vectors may show up in integrals, where we have to remember that they change from pt. to pt.

$$\int_0^{\pi} d\phi\,\hat{\phi} \stackrel{?}{=} \begin{cases} \cancel{\pi\hat{\phi}} \\ \int_0^{\pi} d\phi\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})\Big|_0^{\pi} \end{cases}$$

← Assuming $\hat{\phi}$ is constant gives the wrong answer! What would this answer even mean? $\hat{\phi}$ @ what point?

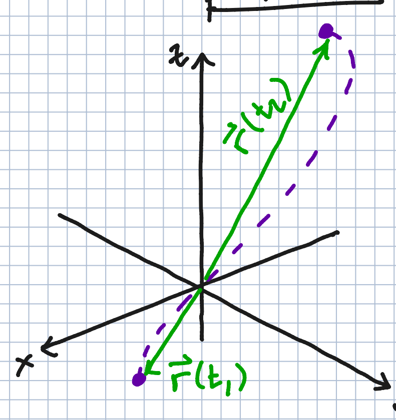
$$= (-1-(1))\,\hat{x} + (0-0)\,\hat{y}$$

$$= -2\,\hat{x} \checkmark$$

- The same goes for <u>derivatives</u>, and this will be especially relevant in your <u>THEORETICAL MECHANICS</u> course!

- As we said on the first day, one reason to use a coord. system is so you can tell me about the motion of some object. In CART. coords you could do this by giving me three functions that specify where it is @ different times: $x(t)$, $y(t)$, and $z(t)$. Then its <u>position</u> is:

$$\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$$



CONSTANT SPEED!!

- For instance, a particle that is moving in a circle in the x-y plane, completing its rotation with period T (so frequency $f = 1/T$) has position:

In x-y plane

$$\vec{r}(t) = R\cos(\omega t + \delta)\,\hat{x} + R\sin(\omega t + \delta) + 0\,\hat{z}$$

Radius

'Angular frequency' $\omega = 2\pi f = \frac{2\pi}{T}$

'Phase' controls where it is @ $t = 0$s. If $\delta = 0$ it starts @ $x(0) = R$ & $y(0) = 0$; if $\delta = \pi/2$ it starts @ $x(0) = 0$, $y(0) = R$, etc.

- Now, once you know the position as a function of time, you can also tell me the object's <u>velocity</u> & <u>acceleration</u>.

$$\vec{v}(t) \equiv \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$\vec{a}(t) \equiv \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y} + \frac{d^2z}{dt^2}\hat{z}$$

NOTICE: Since $\hat{x}, \hat{y}$, & $\hat{z}$ are <u>constant</u>, we don't worry about $\frac{d\hat{x}}{dt}, \frac{d\hat{y}}{dt}, \frac{d\hat{z}}{dt} = 0$!

- For our particle moving in a circle we'd get

$$\vec{v}(t) = -R\omega\sin(\omega t + \delta)\,\hat{x} + R\omega\cos(\omega t + \delta) + 0\,\hat{z}$$

$$|\vec{v}| = \sqrt{R^2\omega^2\sin^2(\omega t + \delta) + R^2\omega^2\cos^2(\omega t + \delta)} = R\omega$$

$$\vec{a}(t) = -R\omega^2\cos(\omega t + \delta)\,\hat{x} - R\omega^2\sin(\omega t + \delta) + 0\,\hat{z}$$

- This all makes sense, but Cart. coords seem like a clumsy way to describe something moving in a circle. Why not use POLAR coords, which have circles (ρ = constant) built in? Using $x = \rho\cos\phi$ & $y = \rho\sin\phi$, get

$$\vec{r}(t) = \underbrace{R}_{\rho = R}\,\hat{\rho} + 0\,\hat{z} \qquad \hat{\rho} = \cos(\underbrace{\omega t + \delta}_{\phi})\,\hat{x} + \sin(\omega t + \delta)\,\hat{y}$$

$\phi \leftarrow \phi = \delta$ @ $t = 0$

- Now here's where we have to be careful! Just looking @ $\vec{r}$, we see $\rho = R$. The info about its motion – the fact that ϕ is changing – is hidden in $\hat{\rho}$.

- In other words, $\hat{\rho}$ depends on ϕ & $\phi = \omega t + \delta$, so $\hat{\rho}$ depends on t as well. When we calculate $\vec{v}$ & $\vec{a}$:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(R\hat{\rho}) = R\frac{d\hat{\rho}}{dt} = R\underbrace{\overbrace{\frac{d\phi}{dt}}^{\omega}\,\frac{d\hat{\rho}}{d\phi}}_{\text{CHAIN RULE!}}$$

$\hat{\phi}$ from our earlier calculation

$$\rightarrow \vec{v} = R\omega\hat{\phi}$$

$$|\vec{v}| = R\omega$$

$$\vec{a} = \frac{d\vec{v}}{dt} = R\omega\frac{d\hat{\phi}}{dt} = R\omega\frac{d\phi}{dt}\overbrace{\frac{d\hat{\phi}}{d\phi}}^{-\hat{\rho}} = R\omega^2(-\hat{\rho})$$

$$\rightarrow \vec{a} = -R\omega^2\hat{\rho}$$

- Once we remembered that the POLAR unit vectors change from pt. to pt., the calc. really was much easier to carry out than it was in CARTESIAN. The right coord system always makes things easier!

- This was a <u>simple</u> example. What about more complicated motion? Suppose we are working in CPC & an object's $\rho, \phi,$ & z are <u>all</u> changing over time.

$$\vec{r}(t) = \rho(t)\underbrace{\hat{\rho}(t)}_{\cos(\phi(t))\,\hat{x} + \sin(\phi(t))\,\hat{y}} + z(t)\hat{z}$$

- To work out $\vec{v}$ & $\vec{a}$, we just need to remember that $\hat{\rho}$ depends on ϕ, and ϕ depends on t:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \frac{dz}{dt}\hat{z} + z\cancelto{0}{\frac{d\hat{z}}{dt}}$$

$$\vec{v} = \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \frac{dz}{dt}\hat{z}$$

$$= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\frac{d\hat{\rho}}{d\phi} + \frac{dz}{dt}\hat{z}$$

$$= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\hat{\phi} + \frac{dz}{dt}\hat{z}$$

$$\hookrightarrow \vec{v} = \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\hat{\phi} + \frac{dz}{dt}\hat{z}$$

Does this make sense? If ρ changes by $d\rho$, that's a dist. $d\rho$ in $\hat{\rho}$ dir, so $v_\rho = d\rho/dt$. If ϕ changes by $d\phi$, distance is $\rho\, d\phi$, so $v_\phi = \rho\, d\phi/dt$. And v_z is the same as Cartesian, of course.

- This is probably a good time to introduce you (if you haven't already seen it) to the 'dot' notation for time derivatives:

$$\dot{f}(t) = \frac{df}{dt} \qquad \ddot{f} = \frac{d^2 f}{dt^2} \qquad \dddot{f} = \frac{d^3 f}{dt^3}, \text{ etc}$$

- So in CPC, the velocity is:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$$

- Likewise, for the acceleration, we have:

$$\vec{a} = \dot{\vec{v}} = \underbrace{\ddot{\rho}\,\hat{\rho} + \dot{\rho}\,\dot{\hat{\rho}}}_{\text{This is just the product rule written in dot notation!}} + \underbrace{\dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\dot{\hat{\phi}}}_{\text{Same here}} + \ddot{z}\hat{z} + \dot{z}\cancelto{0}{\dot{\hat{z}}}$$

$$= \ddot{\rho}\hat{\rho} + \dot{\rho}\overbrace{\dot{\phi}\underbrace{\frac{d\hat{\rho}}{d\phi}}_{\hat{\phi}}}^{\dot{\hat{\rho}}} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\overbrace{\dot{\phi}\underbrace{\frac{d\hat{\phi}}{d\phi}}_{-\hat{\rho}}}^{\dot{\hat{\phi}}} + \ddot{z}\hat{z}$$

$$= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

- To summarize, our expressions for position, velocity, and acceleration in CYLINDRICAL POLAR COORDINATES are:

$$\vec{r} = \rho\,\hat{\rho} + z\,\hat{z}$$

Remember: No ϕ component! Info about ϕ is in $\hat{\rho}$!

$$\vec{v} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\rho} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\,\hat{\phi} + \ddot{z}\,\hat{z}$$

- Things work the same way in any other OCS w/ coordinates q_i & basis vectors $\hat{e}_i$. We just need to know a few things:

  (1) How to write the pos. $\vec{r}$ in terms of the q_i & $\hat{e}_i$. Be careful! This may <u>not</u> be as simple as $q_1\hat{e}_1 + q_2\hat{e}_2 + q_3\hat{e}_3$ — look @ CPC where there is no $\phi\hat{\phi}$ term!

  (2) How to express the basis vectors $\hat{e}_i$ as functions of the coordinates q_i & the Cartesian basis vectors $\hat{x}, \hat{y}, \hat{z}$.

- As an exercise, see if you can work out $\vec{v}$ & $\vec{a}$ for the PARABOLIC CYLINDRICAL COORDS we looked at:

$$(x, y, z) \rightarrow \left(\tfrac{1}{2}(u^2 - v^2),\ uv,\ z\right)$$

Parabolic Coords Example

$x = \frac{1}{2}(u^2 - v^2) \qquad dx = u\,du - v\,dv$

$y = uv \qquad dy = v\,du + u\,dv$

$d\vec{l} = (u\hat{x} + v\hat{y})du + (-v\hat{x} + u\hat{y})dv \quad \Big\} \quad d\vec{l} = du\sqrt{u^2+v^2}\,\hat{u} + dv\sqrt{u^2+v^2}\,\hat{v}$

$h_u = \sqrt{u^2+v^2} \qquad h_v = \sqrt{v^2 + u^2}$

Call them both 'h'!

$\hat{u} = \frac{u\hat{x} + v\hat{y}}{\sqrt{u^2+v^2}} \qquad \hat{v} = \frac{-v\hat{x} + u\hat{y}}{\sqrt{u^2+v^2}}$

$h_u\hat{u} = u\hat{x} + v\hat{y} \qquad h_v\hat{v} = -v\hat{x} + u\hat{y}$

$uh\hat{u} = u^2\hat{x} + uv\hat{y} \qquad vh\hat{v} = -v^2\hat{x} + uv\hat{y}$

$\hookrightarrow (uh\hat{u} - vh\hat{v}) = (u^2+v^2)\hat{x}$

$\hat{x} = \frac{1}{h}(u\hat{u} - v\hat{v})$

$h\hat{u} = \frac{u}{h}(u\hat{u} - v\hat{v}) + v\hat{y}$

$\frac{1}{h}(h^2 - u^2)\hat{u} + \frac{uv}{h}\hat{v} = v\hat{y}$

$\hat{y} = \frac{1}{h}(v\hat{u} + u\hat{v})$

$\vec{r} = x\hat{x} + y\hat{y} = ?$

$\vec{r} = \frac{1}{2}(u^2 - v^2)\frac{1}{h}(u\hat{u} - v\hat{v})$

$\quad + uv\frac{1}{h}(v\hat{u} + u\hat{v})$

$= \frac{1}{h}\left(\frac{1}{2}u^3 + \frac{1}{2}uv^2 + \cancel{uv^2}\right)\hat{u} + \ldots$

$\quad + \frac{1}{h}\left(\frac{1}{2}vu^2 + \frac{1}{2}v^3 + \cancel{u^2v}\right)\hat{v}$

$= \frac{1}{2h}h^2 u\hat{u} + \frac{1}{2h}h^2 v\hat{v}$

$\hookrightarrow \vec{r} = \frac{1}{2}uh\hat{u} + \frac{1}{2}vh\hat{v} \quad \longleftarrow \quad h\hat{u} = u\hat{x} + v\hat{y}, \quad h\hat{v} = -v\hat{x} + u\hat{y}$

$\frac{d}{dt}(h\hat{u}) = \dot{u}\hat{x} + \dot{v}\hat{y} \qquad \frac{d}{dt}(h\hat{v}) = -\dot{v}\hat{x} + \dot{u}\hat{y}$

$= \dot{u}\frac{1}{h}(u\hat{u} - v\hat{v}) \qquad = -\dot{v}\frac{1}{h}(u\hat{u} - v\hat{v})$

$\quad + \dot{v}\frac{1}{h}(v\hat{u} + u\hat{v}) \qquad + \dot{u}\frac{1}{h}(v\hat{u} + u\hat{v})$

$= \frac{1}{h}(u\dot{u} + v\dot{v})\hat{u} \qquad = \frac{1}{h}(-\dot{v}u + \dot{u}v)\hat{u}$

$\quad + \frac{1}{h}(-\dot{u}v + u\dot{v})\hat{v} \qquad + \frac{1}{h}(v\dot{v} + u\dot{u})\hat{v}$

$\dot{\vec{r}} = \frac{1}{2}\dot{u}h\hat{u} + \frac{1}{2}u\frac{d}{dt}(h\hat{u})$

$\quad + \frac{1}{2}\dot{v}h\hat{v} + \frac{1}{2}v\frac{d}{dt}(h\hat{v})$

$= \frac{1}{2}\dot{u}h\hat{u} + \frac{1}{2}u\left(\frac{1}{h}(u\dot{u} + v\dot{v})\hat{u} + \frac{1}{h}(-\dot{u}v + u\dot{v})\hat{v}\right)$

$\quad + \frac{1}{2}\dot{v}h\hat{v} + \frac{1}{2}v\left(\frac{1}{h}(-\dot{v}u + \dot{u}v)\hat{u} + \frac{1}{h}(v\dot{v} + u\dot{u})\hat{v}\right)$

$= \left(\frac{1}{2}\dot{u}h + \frac{1}{2h}(u^2\dot{u} + \cancel{uv\dot{v}}) + \frac{1}{2h}(-\cancel{uv\dot{v}} + v^2\dot{u})\right)\hat{u}$

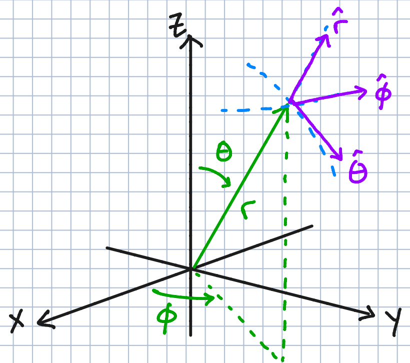
$\quad + \left(\frac{1}{2}\dot{v}h + \frac{1}{2h}(-\cancel{uv\dot{u}} + u^2\dot{v} + v^2\dot{v} + \cancel{uv\dot{u}})\right)\hat{v}$

$= \dot{u}h\hat{u} + \dot{v}h\hat{v} \qquad \Rightarrow \quad \dot{\vec{r}} = \dot{u}h\hat{u} + \dot{v}h\hat{v}$

- Now let's look @ some examples. We've finally got command of some useful math, so let's use it to do some physics!

▨ CELESTIAL MECHANICS

- On HW 3 you will work out the velocity $\dot{\vec{r}}$ & acceleration $\ddot{\vec{r}}$ in SPHERICAL POLAR COORDINATES. Here's what SPC look like:



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

← r is dist. from origin, θ is ∡ down from the z-axis, & ϕ is the ∡ CCW from the x axis in the x-y plane.

$$0 \le r < \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \phi < 2\pi$$

NP @ $\theta = 0$
SP @ $\theta = \pi$
Equator @ $\theta = \pi/2$

ϕ is like longitude. $\phi = 2\pi$ & $\phi = 0$ refer to the same place.

- Your job on the HW will be to <u>derive</u> $\dot{\vec{r}}$ & $\ddot{\vec{r}}$. I won't tell you the full answer, but in the special case where $\theta = \pi/2$ & doesn't change (i.e., a particle that always remains in the x-y plane, so $\dot{\theta} = \ddot{\theta} = 0$) the acceleration is:

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\,\hat{r} + 0\,\hat{\theta} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\,\hat{\phi}$$

This is $\ddot{\vec{r}}$ w/ $\theta = \pi/2$ & $\dot{\theta} = \ddot{\theta} = 0$.

- What can we do with this? Consider a planet orbiting a star. As long as the mass of the star is <u>much</u> larger than the mass of the planet ($M_s \gg M_p$) then the center of mass of the system is basically right @ the center of the star. We'll make this the origin ($r = 0$) of our SPC. To a good approximation the planet orbits around this point.

- (As you know, the star & planet really orbit their COM, which is not quite @ the center of the star. We'll ignore this complication in our first pass @ describing planetary orbits!)

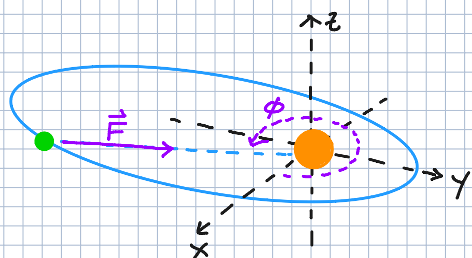
- In <u>mechanics</u> you will show that the orbit always lies in a plane. We can set up our SPC however we like, so let's call the plane of the orbit $\theta = \pi/2$ (i.e., the x-y plane.)

- Newton's Universal Law of Gravitation tells us the force experienced by the planet:

$$\vec{F} = -G\frac{M_s M_p}{r^2}\hat{r}$$

The star is @ $r=0$, so the force on the planet is in the $-\hat{r}$ direction.

The star is @ $r=0$, & the planet is some distance r from the star.



The planet's distance from the star is $r(t)$ & its angular position is $\phi(t)$. Both change over time, but it stays in the $\theta = \pi/2$ plane.

- Now Newton's 2nd Law gives us the EOM for the planet

$$\vec{F} = M_p\vec{a} \Rightarrow M_p(\ddot{r} - r\dot{\phi}^2)\hat{r} + 0\,\hat{\theta} + M_p(2\dot{r}\dot{\phi} + r\ddot{\phi}) = -G\frac{M_s M_p}{r^2}\hat{r}$$

$$\Rightarrow \ddot{r} - r\dot{\phi}^2 = -\frac{GM_s}{r^2} \qquad M_p(2\dot{r}\dot{\phi} + r\ddot{\phi}) = 0$$

$\vec{F}$ had no ϕ component.

- Now we're going to solve these eqns. The next 4 pages are advanced material you'll learn about in THEORETICAL MECHANICS!

- So we've got a pair of coupled, non-linear differential eqns. How do we solve something like this?

- Let's start w/ the ϕ equation, as it looks a little simpler. It might not be immediately apparent, but the ϕ eqn. can be written in terms of a total derivative:

$$M_p(2\dot{r}\dot{\phi} + r\ddot{\phi}) = 0 \Rightarrow \frac{1}{r}\frac{d}{dt}(M_p r^2\dot{\phi}) = 0$$

$$\Rightarrow \frac{d}{dt}(M_p r^2\dot{\phi}) = 0$$

Since $\frac{1}{r}$ is never equal to zero (that'd require $r \to \infty$) the other factor must be 0.

- Since d/dt of $M_p r^2\dot{\phi}$ is zero, it must be that $M_p r^2\dot{\phi}$ is a constant. In fact, it's just the planet's angular momentum. We'll call it J:

$$\boxed{J = M_p r(t)^2\dot{\phi}(t) = \text{constant}}$$

The ϕ EOM was simple b/c angular momentum is conserved. You'll learn how to spot situations like this in MECHANICS.

Both $r(t)$ & $\phi(t)$ will change throughout the orbit, but their product $r^2\dot{\phi}$ will always have the same, constant value.

- (Just so we're being complete, the angular momentum is $\vec{J} = \vec{r} \times \vec{p}$. We know $\vec{r} = r(t)\hat{r}$ & $\vec{p} = M_p \dot{\vec{r}} = M_p(\dot{r}(t)\hat{r} + r(t)\dot{\phi}(t)\hat{\phi})$ when $\theta = \pi/2$, so $\vec{J} = M_p r(t)^2 \dot{\phi}(t)\, \hat{r} \times \hat{\phi} = -M_p r(t)^2 \dot{\phi}(t)\, \hat{\theta}$. In the $\theta = \pi/2$ plane, $\hat{\theta}(\theta = \pi/2) = -\hat{z}$, so $\vec{J} = M_p r(t)^2 \dot{\phi}(t)\, \hat{z}$.)

- Now let's look @ the r eqn. Since M_p shows up on both sides it cancels out & we get:

$$\ddot{r}(t) - r(t)\dot{\phi}(t)^2 = -\frac{GM_s}{r(t)^2}$$

- This looks complicated b/c both r & ϕ appear, but we can use what we learned about J to address this:

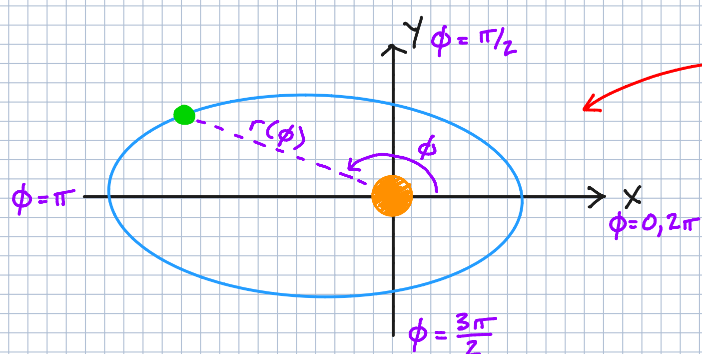
constant!

$$J = M_p r(t)^2 \dot{\phi}(t) \Rightarrow \dot{\phi}(t) = \frac{J}{M_p r(t)^2} = \frac{j}{r(t)^2}$$

$j = J/M_p$. We don't expect M_p to change, so j is also constant!

$$\Rightarrow \boxed{\ddot{r}(t) - \frac{j^2}{r(t)^3} = -\frac{GM_s}{r(t)^2}}$$

- Now how do we solve this? Great question. But first, let me ask you something. Is this really the eqn. you want to solve? If you solve it, you'll know $r(t)$. But if your goal is to figure out the shape of orbits, wouldn't you really rather know $r(\phi)$?



Figuring out $r(\phi)$ – distance from the star as a function of ϕ – seems like a better way of describing the shape of the orbit, right?

- Okay, so how do we do that? We can re-write terms like $\dot{r}(t)$ & $\ddot{r}(t)$ using the chain rule. That is, if we assume r can be written as a function of $\phi(t)$, then:

$$\underbrace{\frac{d}{dt} r(\phi) = \frac{d\phi}{dt}\frac{dr(\phi)}{d\phi}}_{\text{CHAIN RULE}} = \frac{j}{r(\phi)^2}\frac{dr(\phi)}{d\phi}$$

Since $\dot{\phi} = \frac{j}{r^2}$!

- For the $\ddot{r}$ term we need to use the chain rule twice, as well as the product rule. You'll do this on HW 3!

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d\phi}{dt}\frac{d}{d\phi}(\dot{r}) = \frac{d\phi}{dt}\frac{d}{d\phi}\left(\frac{j}{r(\phi)^2}\frac{dr(\phi)}{d\phi}\right) = \ldots$$

- Once we've done this, we arrive @ any eqn. for $r(\phi)$:

$$\frac{j^2}{r(\phi)^4}\frac{d^2r(\phi)}{d\phi^2} - 2\frac{j^2}{r(\phi)^5}\left(\frac{dr(\phi)}{d\phi}\right)^2 - \frac{j^2}{r(\phi)^3} = -\frac{GM_s}{r(\phi)^2}$$

- Wait! Doesn't this look even MORE complicated? Yes, but as is often the case, this is an illusion. If we write the eqn in terms of a new variable it becomes very simple!

$$r(\phi) = \frac{1}{u(\phi)} \Rightarrow \frac{dr(\phi)}{d\phi} = -\frac{1}{u(\phi)^2}\frac{du(\phi)}{d\phi}, \quad \frac{d^2r(\phi)}{d\phi^2} = \ldots$$

$$\Rightarrow \boxed{\frac{d^2u(\phi)}{d\phi^2} + u(\phi) = \frac{GM_s}{j^2}}$$

This is a much simpler equation!

- So, we started with two coupled, non-linear differential eqns for $r(t)$ & $\phi(t)$. But we noticed that one of them just reminded us that angular momentum is conserved for this force, while the other one has turned into a simple-looking eqn. for $u(\phi) = 1/r(\phi)$.

- If we'd tried to write out our eqns. in Cartesian coords we would have been lost! And if we hadn't set up our SPC the right way, things would still be a mess!

THE RIGHT CHOICE OF COORDINATES MAKES EVERYTHING EASIER!

- You are (or will soon be) learning how to solve this sort of eqn. in your Diff. Eq. class.

- I won't derive the sol'n, but if I write it down you can easily check that it works.

$$u(\phi) = \underbrace{c_1 \cos(\phi) + c_2 \sin(\phi)}_{\text{This part satisfies } \frac{d^2u}{d\phi^2} + u = 0} + \underbrace{\frac{M_s G}{j^2}}_{\text{This part has } \frac{d^2}{d\phi^2} = 0, \text{ so when it shows up in } u(\phi) \text{ we get the RHS.}}$$

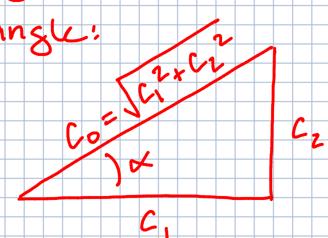
- The eqn. was a second order differential eqn, so its most general sol'n has two unknown constants in it. I called them c_1 & c_2. We can pin them down for a particular planet by giving two pieces of info about its position and/or velocity.

- Now, as you know, planetary orbits are supposed to be ellipses, right? How do we see this?

- First, let me write the constants c_1 & c_2 in a slightly different form:

$$\left.\begin{aligned} c_1 &= c_0 \cos\alpha \\ c_2 &= c_0 \sin\alpha \end{aligned}\right\}$$

I can always do this. I'm just saying that given any two c_1 & c_2, there's a triangle:



- So now the sol'n is:

$$u(\phi) = c_0 \times \underbrace{\left(\cos\phi \cos\alpha + \sin\phi \sin\alpha\right)}_{\cos(\phi - \alpha)} + \frac{G M_s}{j^2}$$

Check that GM_s/j^2 has units of 1/length. c_0 has same units!

$r = \frac{1}{u}$

$$r(\phi) = \frac{1}{c_0 \cos(\phi - \alpha) + \frac{M_s G}{j^2}}$$

$$\Rightarrow \boxed{r(\phi) = \frac{\left(\frac{j^2}{M_s G}\right)}{1 + \varepsilon \cos(\phi - \alpha)}}$$

The numerator has units of length. The combination $\varepsilon = c_0 j^2/(M_s G)$ is a number with no units.

- What exactly have we shown? First, the shape of the orbit is characterized by three quantities. One of these is the ratio $j^2/(M_s G)$, which has units of length. Then there are two plain numbers: the coeff. ε of the cos in the denominator, and the α that shows up inside the cos.

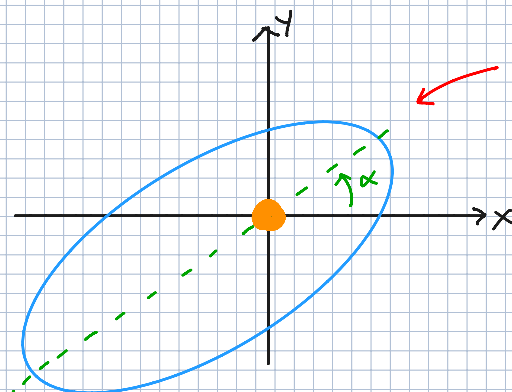
$$r(\phi) = \frac{l}{1+\varepsilon\cos(\phi-\alpha)}$$

This is $j^2/(M_s G)$

- As expected, when $0<\varepsilon<1$ this is an ELLIPSE w/ semi-major axis a & semi-minor axis b, tilted @ angle α:

$$a = \frac{j^2}{M_s G} \times \frac{1}{1-\varepsilon^2}$$

$$b = \frac{j^2}{M_s G} \times \frac{1}{\sqrt{1-\varepsilon^2}}$$



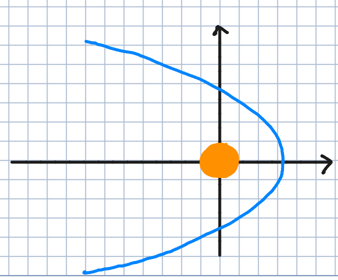
Star is @ one of the two foci.

When $\varepsilon=0$, $a=b=R$ & the orbit is a circle. In that case $j = R^2\dot\phi = Rv$ ($v = R\dot\phi$ for UCM) &

$$R = \frac{R^2v^2}{M_s G} \Rightarrow \frac{M_p v^2}{R} = G\frac{M_s M_p}{R^2}$$

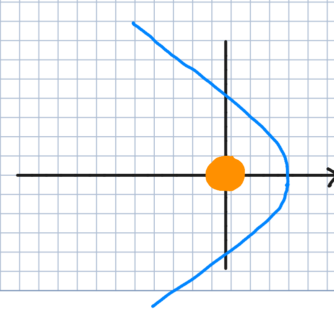
Neptune's moon Triton has the lowest eccentricity of any orbit in the solar system: $\varepsilon = 0.000016$.

- But this also describes other sorts of orbits. When $\varepsilon=1$ our formula gives a parabola, and if $\varepsilon>1$ we get a hyperbola. These are the orbits of an object with just enough or more than enough (respectively) velocity to escape the gravitational attraction of the star.



$\varepsilon=1$

Some comets, captured objects, etc.



$\varepsilon>1$