Velocity & Acceleration in OCS

- Once we know the scale factors & unit vectors for an OCS, working out $\vec{v}$ & $\vec{a}$ is straightforward. (Though it might be a little tedious!)
- As an example, let's consider the "3 dimensional Parabolic Coordinates" (u, v, ϕ) defined by

$$x = uv\cos\phi \qquad y = uv\sin\phi \qquad z = \tfrac{1}{2}u^2 - \tfrac{1}{2}v^2$$

$$\text{w/ } 0 \le u, v < \infty \ , \ 0 \le \phi < 2\pi$$

- Surfaces of const. u are concave-down paraboloids, const. v surfaces are concave-up paraboloids, & const. ϕ surfaces are planes.
- First let's find the scale factors & unit vectors

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = uv\cos\phi\,\hat{x} + uv\sin\phi\,\hat{y} + \left(\tfrac{1}{2}u^2 - \tfrac{1}{2}v^2\right)\hat{z}$$

$$\frac{\partial\vec{r}}{\partial u} = v\cos\phi\,\hat{x} + v\sin\phi\,\hat{y} + u\,\hat{z} = h_u\,\hat{u}$$

$$\rightarrow h_u = \left(v^2\cos^2\phi + v^2\sin^2\phi + u^2\right)^{1/2} = \sqrt{u^2+v^2}$$

$$\hat{u} = \frac{v\cos\phi}{\sqrt{u^2+v^2}}\,\hat{x} + \frac{v\sin\phi}{\sqrt{u^2+v^2}}\,\hat{y} + \frac{u}{\sqrt{u^2+v^2}}\,\hat{z}$$

$$\frac{\partial\vec{r}}{\partial v} = u\cos\phi\,\hat{x} + u\sin\phi\,\hat{y} - v\,\hat{z} = h_v\,\hat{v}$$

$$\rightarrow h_v = \left(u^2\cos^2\phi + u^2\sin^2\phi + v^2\right)^{1/2} = \sqrt{u^2+v^2}$$

$$\hat{v} = \frac{u\cos\phi}{\sqrt{u^2+v^2}}\,\hat{x} + \frac{u\sin\phi}{\sqrt{u^2+v^2}}\,\hat{y} - \frac{v}{\sqrt{u^2+v^2}}\,\hat{z}$$

$$\frac{\partial\vec{r}}{\partial\phi} = -uv\sin\phi\,\hat{x} + uv\cos\phi\,\hat{y} = h_\phi\,\hat{\phi}$$

$$\rightarrow h_\phi = \left(u^2v^2\sin^2\phi + u^2v^2\cos^2\phi\right)^{1/2} = uv$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

- Now, how do we express $\vec{r}$ in this coordinate system? We get the $u, v,$ & ϕ components of any vector by evaluating its dot product w/ $\hat{u}, \hat{v},$ & $\hat{\phi}$.

$$r_u = \vec{r}\cdot\hat{u} = \left(uv\cos\phi\,\hat{x} + uv\sin\phi\,\hat{y} + \left(\tfrac{1}{2}u^2 - \tfrac{1}{2}v^2\right)\hat{z}\right)\cdot\left(\frac{v\cos\phi\,\hat{x} + v\sin\phi\,\hat{y} + u\hat{z}}{\sqrt{u^2+v^2}}\right)$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(\overbrace{uv^2\cos^2\phi + uv^2\sin^2\phi}^{uv^2} + \tfrac{1}{2}u^3 - \tfrac{1}{2}uv^2\right)$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(\tfrac{1}{2}u^3 + \tfrac{1}{2}uv^2\right) = \tfrac{1}{2}u\sqrt{u^2+v^2}$$

$$r_v = \vec{r}\cdot\hat{v} = \left(uv\cos\phi\,\hat{x} + uv\sin\phi\,\hat{y} + \left(\tfrac{1}{2}u^2 - \tfrac{1}{2}v^2\right)\hat{z}\right)\cdot\left(\frac{u\cos\phi\,\hat{x} + u\sin\phi\,\hat{y} - v\hat{z}}{\sqrt{u^2+v^2}}\right)$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(\overbrace{u^2v\cos^2\phi + u^2v\sin^2\phi}^{u^2v} - \tfrac{1}{2}u^2v + \tfrac{1}{2}v^3\right)$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(\tfrac{1}{2}u^2v + \tfrac{1}{2}v^3\right) = \tfrac{1}{2}v\sqrt{u^2+v^2}$$

$$r_\phi = \vec{r}\cdot\hat{\phi} = \left(uv\cos\phi\,\hat{x} + uv\sin\phi\,\hat{y} + \left(\tfrac{1}{2}u^2 - \tfrac{1}{2}v^2\right)\hat{z}\right)\cdot\left(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right)$$

$$= -uv\cos\phi\sin\phi + uv\sin\phi\cos\phi = 0$$

$$\hookrightarrow \boxed{\vec{r} = \tfrac{1}{2}u\sqrt{u^2+v^2}\,\hat{u} + \tfrac{1}{2}v\sqrt{u^2+v^2}\,\hat{v}}$$

$\vec{r}$ has no ϕ-component, just like in cylindrical polar coords. All the info about ϕ is in $\hat{u}$ & $\hat{v}$.

- This is how we express $\vec{r}$ in <u>any</u> orthogonal coordinate system: Start w/ $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$; express $x, y,$ & z in terms of $q_1, q_2,$ & q_3; find the scale factors h_1, h_2, h_3 & unit vectors $\hat{e}_1, \hat{e}_2,$ & $\hat{e}_3$; then work out the components of $\vec{r}$ by taking the dot products $\hat{e}_1\cdot\vec{r}$, $\hat{e}_2\cdot\vec{r}$, and $\hat{e}_3\cdot\vec{r}$.

- Remember that some of the components of $\vec{r}$ may be zero! In CPC, $\vec{r} = \rho\hat{\rho} + z\hat{z}$. In SPC it's $\vec{r} = r\hat{r}$. Here, for these parabolic coordinates we found $\hat{u}$ & $\hat{v}$ components but no $\hat{\phi}$ component.

- So now that we have $\vec{r}$, how do we find the velocity? There are multiple ways to do this. If you are describing the position of a moving object then $u, v, \text{ \& } \phi$ will all be functions of time. So you could just take the t derivative of $\vec{r}$:

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\frac{1}{2}u\sqrt{u^2+v^2}\,\hat{u} + \frac{1}{2}v\sqrt{u^2+v^2}\,\hat{v}\right)$$

- However, you need to remember that $\hat{u}$ & $\hat{v}$ depend on $u, v,$ & ϕ, which are all functions of t when we're talking about a moving object.

$$\frac{d}{dt}(\hat{u}) = \frac{\partial u}{\partial t}\frac{\partial \hat{u}}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial \hat{u}}{\partial v} + \frac{\partial \phi}{\partial t}\frac{\partial \hat{u}}{\partial \phi} = \dot{u}\frac{\partial \hat{u}}{\partial u} + \dot{v}\frac{\partial \hat{u}}{\partial v} + \dot{\phi}\frac{\partial \hat{u}}{\partial \phi}$$

Working out d/dt of $\hat{u}, \hat{v},$ & $\hat{\phi}$ may be complicated.

- Another way to calculate the velocity in an OCS is to start w/ $d\vec{r}/dt = \dot{x}\,\hat{x} + \dot{y}\,\hat{y} + \dot{z}\,\hat{z}$ in Cartesian coords; work out $\dot{x}, \dot{y},$ & $\dot{z}$ in terms of q_1, q_2, q_3 & $\dot{q}_1, \dot{q}_2, \dot{q}_3$; and then use dot products to find the $\hat{e}_1, \hat{e}_2,$ & $\hat{e}_3$ components.

- But probably the <u>easiest</u> way to find $d\vec{r}/dt$ in an orthogonal coordinate system is to remember that $\vec{r}$ — the position — is just a (vector) function of 3 coords $q_1, q_2,$ & q_3. If the object's pos. is changing then $q_1, q_2,$ & q_3 are functions of t. So:

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$$\frac{d}{dt}(\vec{r}) = \frac{\partial q_1}{\partial t}\underbrace{\frac{\partial \vec{r}}{\partial q_1}}_{h_1\hat{e}_1} + \frac{\partial q_2}{\partial t}\underbrace{\frac{\partial \vec{r}}{\partial q_2}}_{h_2\hat{e}_2} + \frac{\partial q_3}{\partial t}\underbrace{\frac{\partial \vec{r}}{\partial q_3}}_{h_3\hat{e}_3}$$

$$\Rightarrow \boxed{\frac{d\vec{r}}{dt} = \dot{q}_1 h_1 \hat{e}_1 + \dot{q}_2 h_2 \hat{e}_2 + \dot{q}_3 h_3 \hat{e}_3}$$

- This a general result that is true for <u>any</u> OCS. It's just the statement that $\vec{r}$ is a function of q_1, q_2, q_3, and $\partial\vec{r}/\partial q_i = h_i \hat{e}_i$. Another way of looking @ it is that a small change in q_i moves you a distance $h_i dq_i$ in the $\hat{e}_i$ direction. So if the coords change a small amount this displaces the object (changes its position) by

$$d\vec{r} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3$$

Dividing this by dt to get the <u>rate</u> of displacement gives the result for $d\vec{r}/dt$.

- So, for the 3-D parabolic coordinates, the velocity is

$$\frac{d\vec{r}}{dt} = \dot{u}\sqrt{u^2+v^2}\,\hat{u} + \dot{v}\sqrt{u^2+v^2}\,\hat{v} + \dot{\phi}\,uv\,\hat{\phi}$$

Any one of the calculations described above will (eventually) lead to this result.

- Now what about the acceleration $d^2\vec{r}/dt^2$? Again, there are different ways of approaching this. For instance, we could start in Cartesian w/ $d^2\vec{r}/dt^2 = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$; use what we know about the OCS to express $\ddot{x}, \ddot{y}, \& \ddot{z}$ in terms of q_1, q_2, q_3 & their time derivatives, & then evaluate dot products to get the $\hat{e}_1, \hat{e}_2, \& \hat{e}_3$ components.

- Or we could just take d/dt of the velocity:

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\dot{u}\sqrt{u^2+v^2}\,\hat{u} + \dot{v}\sqrt{u^2+v^2}\,\hat{v} + \dot{\phi}\,uv\,\hat{\phi}\right)$$

- Some of these derivatives are easy to evaluate; others require more work.

$$\frac{d}{dt}\left(\dot{u}\sqrt{u^2+v^2}\,\hat{u}\right) = \ddot{u}\sqrt{u^2+v^2}\,\hat{u} + \dot{u}\frac{1}{2\sqrt{u^2+v^2}}(2u\dot{u}+2v\dot{v})\,\hat{u} + \dot{u}\sqrt{u^2+v^2}\,\frac{d\hat{u}}{dt}$$

- What's that last term? It involves d/dt of $\hat{u}$.

$$\frac{d\hat{u}}{dt} = \frac{d}{dt}\left(\frac{v\cos\phi}{\sqrt{u^2+v^2}}\,\hat{x} + \frac{v\sin\phi}{\sqrt{u^2+v^2}}\,\hat{y} + \frac{u}{\sqrt{u^2+v^2}}\,\hat{z}\right)$$

This changes over time! Unlike $\hat{x}, \hat{y}, \& \hat{z}$, derivatives of the unit vectors for an OCS usually aren't zero.

$$\frac{d\hat{u}}{dt} = \dot{u}\frac{\partial\hat{u}}{\partial u} + \dot{v}\frac{\partial\hat{u}}{\partial v} + \dot{\phi}\frac{\partial\hat{u}}{\partial\phi}$$

(LOTS OF CALC & ALGEBRA)

$$\frac{d\hat{u}}{dt} = \frac{(u\dot{v}-v\dot{u})}{u^2+v^2}\,\hat{v} + \frac{v\dot{\phi}}{\sqrt{u^2+v^2}}\,\hat{\phi}$$

$$\frac{d\hat{v}}{dt} = \frac{(v\dot{u}-u\dot{v})}{u^2+v^2}\,\hat{v} + \frac{u\dot{\phi}}{\sqrt{u^2+v^2}}\,\hat{\phi}$$

$$\frac{d\hat{\phi}}{dt} = -\frac{v\dot{\phi}}{\sqrt{u^2+v^2}}\,\hat{u} - \frac{u\dot{\phi}}{\sqrt{u^2+v^2}}\,\hat{v}$$

NOTICE: $d\hat{u}/dt$ has no $\hat{u}$ component. This makes sense, because $\hat{u}\cdot\hat{u}=1$

$$\rightarrow \underbrace{\frac{d}{dt}(\hat{u}\cdot\hat{u})}_{=0 \text{ b/c } \hat{u}\cdot\hat{u}=1} = 2\,\underbrace{\hat{u}\cdot\frac{d\hat{u}}{dt}}_{\hat{u} \text{ component of } d\hat{u}/dt}$$

- Anyway, just remember that you have to carefully evaluate d/dt of unit vectors in an OCS, because unlike $\hat{x}, \hat{y}, \hat{z}$ the derivatives of $\hat{e}_1, \hat{e}_2, \hat{e}_3$ usually aren't zero.

- For 3-D Parabolic Coordinates, a long-ish calculation gives

$$\frac{d^2\vec{r}}{dt^2} = \left(\ddot{u}\sqrt{u^2+v^2} + \frac{u\dot{u}^2-u\dot{v}^2-uv^2\dot{\phi}^2+2v\dot{u}\dot{v}}{\sqrt{u^2+v^2}}\right)\hat{u}$$
$$+\left(\ddot{v}\sqrt{u^2+v^2} + \frac{v\dot{v}^2-v\dot{u}^2-vu^2\dot{\phi}^2+2u\dot{u}\dot{v}}{\sqrt{u^2+v^2}}\right)\hat{v}$$
$$+\left(uv\ddot{\phi} + 2u\dot{v}\dot{\phi} + 2v\dot{u}\dot{\phi}\right)\hat{\phi}$$