Method of Frobenius – Examples

(1) $x\,y''(x) + 2y'(x) + x\,y(x) = 0$

Assume: $y(x) = \sum_{n=0} c_n x^{n+s}$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1)\,c_n x^{n+s-1} + \sum_{n=0}^{\infty} 2(n+s)\,c_n x^{n+s-1} + \sum_{n=0}^{\infty} c_n x^{n+s+1} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} (n+s)(n+s+1)\,c_n x^{n+s-1}}_{\text{The } n=0 \text{ \& } n=1 \text{ terms here do not appear in the other sum.}} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+s+1}}_{\text{First term is } x^{s+1}\text{, then } x^{s+2}\text{, etc.}} = 0$$

$$\Rightarrow 0 = s(s+1)\,c_0 x^{s-1} + (s+1)(s+2)\,c_1 x^{s} + \sum_{n=2}^{\infty} (n+s)(n+s+1)\,c_n x^{n+s-1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+s+1}}_{\sum_{n-2=0}^{\infty} c_{n-2} x^{n+s-1}}$$

$$\Rightarrow 0 = s(s+1)\,c_0 x^{s-1} + (s+1)(s+2)\,c_1 x^{s} + \sum_{n=2}^{\infty}\left[(n+s)(n+s+1)c_n + c_{n-2}\right] x^{n+s-1}$$

Indicial Eqn: $s(s+1) = 0$ $\Rightarrow s = -1$ or $s = 0$

First consider $s=-1$. I don't want to mix up my $s=-1$ & $s=0$ calculations, so I'll use 'a_n' for the coefficients of my generalized power series sol'n in this case:

$$s=-1 \Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^{n-1} = a_0 x^{-1} + a_1 + a_2 \cdot x + a_3 x^2 + \ldots$$

$$\Rightarrow 0 = 0 \cdot a_0 \cdot x^{-2} + 0 \cdot a_1 \cdot x^{-1} + \sum_{n=2}^{\infty}\left[n(n-1)\,a_n + a_{n-2}\right] x^{n-2}$$

a_0 & a_1 are both arbitrary – any values work.

Recurrence: $n(n-1)a_n + a_{n-2} = 0 \Rightarrow a_n = -\dfrac{a_{n-2}}{n(n-1)}$, $n \geq 2$

$\rightarrow a_2 = -\frac{1}{2}a_0$, $a_3 = -\frac{1}{6}a_1$, $a_4 = -\dfrac{a_2}{12} = +\dfrac{a_0}{4!}$, $a_5 = -\dfrac{a_3}{20} = +a_1/5!$

So for $s=-1$ we get two independent solutions:

$$y(x) = \frac{1}{x} a_0 \times \left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \dots\right) + \frac{1}{x} a_1 \cdot \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots\right)$$

Next, let's consider $s=0$. I'll stick w/ c_n for the coefficients in my power series here.

$$y(x) = \cancel{x^0}^{\,1} \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

$$\Rightarrow 0 = \underbrace{0 \cdot c_0 \cdot x^{-1}}_{c_0 \text{ undetermined}} + \underbrace{2 c_1 x^0}_{c_1 = 0} + \sum_{n=2}^{\infty} \underbrace{\left[n(n+1) c_n + c_{n-2} \right]}_{} x^{n-1}$$

RECURRENCE: $n(n+1)c_n + c_{n-2} = 0$

$$c_n = -\frac{c_{n-2}}{n(n+1)} \Rightarrow c_2 = -\frac{c_0}{6}, \quad c_4 = -\frac{c_2}{20} = +\frac{c_0}{5!}, \quad c_6 = -\frac{c_4}{42} = +\frac{c_0}{7!}$$

$$c_3 = -\frac{c_1}{12} = 0, \quad c_5 = c_7 = \dots = 0 \quad (\text{All } c_{2k+1} = 0)$$

So the $s=0$ sol'n is

$$y(x) = c_0 \times \left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \dots\right)$$

The 2 roots of the indicial eqn differed by an integer, so this isn't surprising.

But notice that this is one of the sol'ns we already found in the $s=-1$ case!

$$y(x) = \frac{c_0}{x} \times \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots\right)$$

Mutiplying by $\frac{x}{x}$ shows this is the same as our a_1 sol'n.

The general sol'n of the ODE is then:

$$y(x) = a_0 \times \left(\frac{1}{x} - \frac{1}{2}x + \frac{1}{4!}x^3 + \dots\right) + a_1 \times \left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \dots\right)$$

Which can also be written as $a_0 \times \frac{\cos x}{x} + a_1 \times \frac{\sin x}{x}$.

(2) $3x^2 y''(x) + x\cdot(1+x)\,y'(x) - y(x) = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+s}$$

$$\rightarrow\ 0 = \sum_{n=0}^{\infty} 3(n+s)(n+s-1)\,c_n\,x^{n+s} + \sum_{n=0}^{\infty}\left[(n+s)c_n x^{n+s} + (n+s)c_n x^{n+s+1}\right] - \sum_{n=0}^{\infty} c_n x^{n+s}$$

$$= \sum_{n=0}^{\infty}\left[3(n+s)(n+s-1) + (n+s) - 1\right] c_n\,x^{n+s} + \sum_{n=0}^{\infty}(n+s)\,c_n\,x^{n+s+1}$$

The 1st sum contains powers $x^s, x^{s+1}, x^{s+2}, \ldots$, while the 2nd sum has powers $x^{s+1}, x^{s+2}, \ldots$. So:

$$0 = \left[3s(s-1)+s-1\right] c_0\,x^s + \sum_{n=1}^{\infty}\left[3(n+s)(n+s-1) + (n+s) - 1\right] c_n\,x^{n+s}$$
$$+ \sum_{n-1=0}^{\infty}(n-1+s)\,c_{n-1}\,x^{n+s}$$

Re-indexed 2nd sum $(n \rightarrow n-1)$

$$= \underbrace{(3s+1)(s-1)}\,c_0\,x^s + \sum_{n=1}^{\infty}\Big[\underbrace{(3n+3s+1)(n+s-1)\,c_n + (n+s-1)\,c_{n-1}}\Big]\,x^{n+s}$$

INDICIAL EQN.

$$(3s+1)(s-1) = 0$$

RECURRENCE REL'N

$$(3n+3s+1)(n+s-1)\,c_n + (n+s-1)\,c_{n-1} = 0$$

The indicial eqn. gives $s = -1/3$ and $s = 1$. Let's look @ the $s = -1/3$ sol'n first:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n-1/3} = a_0 x^{-1/3} + a_1 x^{2/3} + a_2 x^{5/3} + \ldots$$

$$(3n + 3\cdot(-1/3) + 1)(n + (-1/3) - 1)\,a_n + (n + (-1/3) - 1)\,a_{n-1} = 0$$

$$\rightarrow (n - 4/3)\times\left[\,3n\,a_n + a_{n-1}\right] = 0 \implies a_n = \frac{-a_{n-1}}{3n}$$

$$a_1 = -\frac{1}{3}a_0 \qquad a_2 = -\frac{1}{6}a_1 = \frac{1}{18}a_0 \qquad a_3 = -\frac{1}{9}a_2 = -\frac{1}{162}a_0$$

$$\hookrightarrow y(x) = a_0\times\left(x^{-1/3} - \frac{1}{3}x^{2/3} + \frac{1}{18}x^{5/3} - \frac{1}{162}x^{8/3} + \ldots\right)$$

This sol'n starts @ $x^{-1/3}$, and proceeds in powers of x.

Now lets consider the $s=1$ sol'n.

$$y(x) = \sum_{n=0}^{\infty} b_n x^{n+1} = b_0 \cdot x + b_1 x^2 + b_2 x^3 + \ldots$$

$$\hookrightarrow (3n+3\cdot 1+1)(n+1-1)\, b_n + (n+1-1)\cdot b_{n-1} = 0$$

$$\Longrightarrow \quad b_n = -\frac{b_{n-1}}{3n+4}$$

$$b_1 = -\frac{b_0}{7} \qquad b_2 = -\frac{b_1}{10} = \frac{b_0}{70} \qquad b_3 = -\frac{b_2}{13} = -\frac{b_0}{910}$$

$$\hookrightarrow y(x) = b_0 \times \left(x - \frac{1}{7} x^2 + \frac{1}{70} x^3 - \frac{1}{910} x^4 + \ldots \right)$$

Unlike the last example, the difference of the roots of the indicial eqn. is not an integer. So each value of s gives us a distinct sol'n.

$$y(x) = a_0 \times \left(x^{-1/3} - \frac{1}{3} x^{2/3} + \frac{1}{18} x^{5/3} - \frac{1}{162} x^{8/3} + \ldots \right)$$
$$+ b_0 \times \left(x - \frac{1}{7} x^2 + \frac{1}{70} x^3 - \frac{1}{910} x^4 + \ldots \right)$$