- In class we saw that observers in two different frames of reference may disagree on things like charge densities, currents, and fields. How does this work?
- First, let's remind ourselves how our spacetime description of an event changes between reference frames.
- Suppose I am in an “inertial” reference frame. That is, Newton's Laws accurately describe my observations about Forces, accelerations, etc.
- I will describe where things happen using a coordinate system w/me @ its origin, and when they happen by reading a time off my watch.
- Meanwhile, you zip past me moving w/ constant velocity \( \mathbf{v} = v \mathbf{u} \). We are briefly in the same place as you pass by, and we sync our watches @ that instant so they both read ‘0’.
- So as a result, we agree on the ‘where’ and ‘when’ of that event. I say that you passed me @ \( \mathbf{r} = 0 \) \& \( t = 0 \) moving w/ velocity \( \mathbf{v} = v \mathbf{u} \). You say that I passed you @ \( \mathbf{r}' = 0 \) \& \( t' = 0 \) moving w/ velocity \(-v \mathbf{u}\).
- Now suppose I notice some event that happens @ position \( \mathbf{r}' \), \& time \( t'_i \). You will assign different spacetime coords to that event.
- First, let me orient my axes so that one of them is the dir. \( \mathbf{u} \) I see you moving in.
That is:
\[ \mathbf{r}_i = \mathbf{r}_{i,\|} \hat{\mathbf{u}} + \mathbf{r}_{i,\perp} \quad \text{with} \quad \mathbf{r}_{i,\perp} \cdot \hat{\mathbf{u}} = 0 \]

So if you were moving in what I called the \( \hat{x} \) direction, then \( \mathbf{r}_{i,\|} \) would be \( x_i \) \& \( \mathbf{r}_{i,\perp} \) would be \( y, y + z, \hat{z} \). But this lets us write our results more generally.

I can write \( \mathbf{r}_i \) as some distance \( r_{i,\|} \) along the direction \( \hat{\mathbf{u}} \) of your velocity, and then a vector \( \mathbf{r}_{i,\perp} \) in a plane that is \( \perp \) to \( \hat{\mathbf{u}} \).

In your frame of reference, you will describe this event differently. You'll agree with my \( \mathbf{r}_{i,\perp} \) but not \( t_i \) or \( r_{i,\|} \):

\[
\begin{align*}
 t_i' &= \tilde{\tau}(w) \cdot (t_i - \beta(w) \frac{1}{c} r_{i,\|}) \\
 r_{i,\|}' &= \tilde{\tau}(w) \cdot (r_{i,\|} - \beta(w) c t_i) \\
 r_{i,\perp}' &= \mathbf{r}_{i,\perp}
\end{align*}
\]

This is the same Lorentz transformation you've seen before, written in a way that leaves the direction of \( \hat{\mathbf{u}} \) arbitrary. (Check that this looks like what you expect for \( \tilde{\tau} = \gamma u \).)

Now, suppose I see a collection of charge \& current. I'll describe them in terms of a charge density \( \hat{\mathbf{j}} \) and a current density \( \mathbf{J} \), which I'll write as:

\[ \mathbf{J} = \mathbf{J}_{\|} \hat{\mathbf{u}} + \mathbf{J}_{\perp} \quad \text{with} \quad \mathbf{J}_{\perp} \cdot \hat{\mathbf{u}} = 0 \]
- You will describe the same collection of charges & currents differently. That makes sense, something I say is @ rest will appear to be moving w/ vel. -\( \mathbf{u} \hat{\mathbf{u}} \) in your frame of reference.

\[
\begin{align*}
\rho' &= \sigma(u) \cdot \left( \rho - \beta(u) \frac{1}{c} J_{\parallel} \right) \\
J'_{\parallel} &= \gamma(u) \cdot \left( J_{\parallel} - \beta(u) c \rho \right) \\
J'_{\perp} &= J_{\perp}
\end{align*}
\]

- This also applies to a line charge density \( \lambda \) & a current \( \mathbf{I} = I_{\parallel} \hat{\mathbf{u}} + I_{\perp} \)

\[
\begin{align*}
\lambda' &= \sigma(u) \cdot \left( \lambda - \beta(u) \frac{1}{c} I_{\parallel} \right) \\
I'_{\parallel} &= \gamma(u) \cdot \left( I_{\parallel} - \beta(u) c \lambda \right) \\
I'_{\perp} &= I_{\perp}
\end{align*}
\]

- So, suppose I see a line charge density \( \lambda \) consisting of a row of evenly spaced charges, but no current (\( \mathbf{I} = 0 \)). You would interpret that as

\[
\lambda' = \sigma(u) \cdot \frac{\lambda}{\sqrt{1 - \mathbf{u} \cdot \mathbf{u} - \frac{\lambda}{c^2}}} \quad \text{From your Provi, charges are moving w/ vel. } -\mathbf{u} \hat{\mathbf{u}} \text{. They appear to be spaced closer together b/c of Length CONTRACTION.}
\]

\[
\begin{align*}
I'_{\parallel} &= -\sigma(u) \beta(u) c \lambda = -\frac{\mathbf{u} \cdot \lambda}{\sqrt{1 - \mathbf{u} \cdot \mathbf{u} - \frac{\lambda}{c^2}}} \quad \text{A line charge } \lambda' \text{ moving w/ velocity } -\mathbf{u} \hat{\mathbf{u}} \text{ gives a current } -\lambda' \mathbf{u} \hat{\mathbf{u}}.
\end{align*}
\]

- Likewise, if I saw a line charge \( \lambda \) moving w/ vel. \( \mathbf{u} \hat{\mathbf{u}} \) (so \( \mathbf{I} = \lambda \mathbf{u} \hat{\mathbf{u}} \)) you would perceive that as a stationary charge density:

\[
\begin{align*}
\lambda' &= \sigma(u) \cdot \left( \lambda - \beta(u) \frac{\lambda}{c^2} \right) = \frac{1}{\sqrt{1 - \mathbf{u} \cdot \mathbf{u} - \frac{\lambda}{c^2}}} \left( 1 - \frac{\mathbf{u} \cdot \lambda}{c^2} \right) \lambda = \sqrt{1 - \mathbf{u} \cdot \mathbf{u} - \frac{\lambda}{c^2}} \lambda \\
I'_{\parallel} &= \sigma(u) \cdot \left( \lambda \mathbf{u} + \beta(u) c \lambda \right) = 0 \\
I'_{\perp} &= 0
\end{align*}
\]
- Since you & I tend to disagree on charge & current densities, we naturally disagree on the fields they produce.

- If I say that a collection of charges & currents produce an electric field $\vec{E}$ & magnetic field $\vec{B}$, then you see different fields $\vec{E}'$ & $\vec{B}'$.

- Let me write $\vec{E}$ & $\vec{B}$ as

\[
\vec{E} = E_\parallel \hat{u} + E_\perp \hat{v}, \quad \vec{B} = B_\parallel \hat{u} + B_\perp \hat{v}
\]

- $E_\parallel = E \cdot \hat{u}$ is the part of $\vec{E}$ in the direction of your vel. velocity: $E_\parallel \hat{u} = 0$

- Then the fields you see are

\[
\begin{align*}
\vec{E}' &= E_\parallel \hat{u} + \gamma(u) E_\perp \hat{v} + \gamma(u) \hat{v} \times B_\perp \\
\vec{B}' &= B_\parallel \hat{u} + \gamma(u) B_\perp - \gamma(u) \frac{1}{c^2} \hat{v} \times E_\perp 
\end{align*}
\]

- Notice that we agree on the components of the fields in the direction $\hat{u}$. It's the parts perp. to $\hat{u}$ that we disagree on.

- Let's be a bit more explicit. Imagine that we have two unit vectors $\hat{v} \times \hat{w}$ that are $\perp$ to $\hat{w}$ & to each other. Together, $\hat{u}, \hat{v}, \hat{w}$ make up a right-handed coord. system $\hat{w}$

\[
\hat{u} \times \hat{v} = \hat{w}, \quad \hat{v} \times \hat{w} = \hat{u}, \quad \hat{w} \times \hat{u} = \hat{v}
\]

- The components of $\vec{E}_\perp$ & $\vec{B}_\perp$ can be written

\[
\vec{E}_\perp = E_v \hat{v} + E_w \hat{w}, \quad \vec{B}_\perp = B_v \hat{v} + B_w \hat{w}
\]

\[
\Rightarrow \hat{u} \times \vec{E}_\perp = -u E_w \hat{v} + u E_v \hat{w}, \quad \hat{u} \times \vec{B}_\perp = -u B_w \hat{v} + u B_v \hat{w}
\]
So if I observe fields

\[ E = E_u \hat{u} + E_v \hat{v} + E_w \hat{w} \quad B = B_u \hat{u} + B_v \hat{v} + B_w \hat{w} \]

Then the fields you measure are

\[ E_u' = E_u \quad E_v' = \gamma(u) \left( E_v - uB_w \right) \quad E_w' = \gamma(u) \left( E_w + uB_v \right) \]
\[ B_u' = B_u \quad B_v' = \gamma(u) \left( B_v + \frac{u}{c^2} E_w \right) \quad B_w' = \gamma(u) \left( B_w - \frac{u}{c^2} E_v \right) \]

Does this agree w/ what we saw in class? We considered an electrically neutral wire carrying a current \( I = \lambda u \hat{z} \). It produced no electric field, & a magnetic field \( \hat{B} = (\mu_0 I/2\pi s) \hat{\phi} \). In a frame of reference moving w/ \( \hat{u} = u \hat{z} \), you'd see:

\[ \hat{u} = \hat{e} \quad \hat{v} = \hat{s} \quad \hat{w} = \hat{\phi} \quad \vec{E} = 0 \quad \vec{B} = \frac{\mu_0 E}{2\pi s} \hat{n} = \frac{\mu_0 \lambda u}{2\pi s} \hat{n} \]
\[ E_u' = 0 \quad E_s' = \gamma(u) \left( 0 - \frac{\mu_0 \lambda u}{2\pi s} \right) \quad E_{\phi}' = 0 \]
\[ B_u' = 0 \quad B_s' = 0 \quad B_{\phi}' = \gamma(u) \left( \frac{\mu_0 \lambda u}{2\pi s} - 0 \right) \]
\[ \hat{E}' = -\frac{\mu_0}{2\pi s} \frac{\lambda u \hat{V}}{\sqrt{1 - u^2/c^2}} \hat{s} \quad \hat{B}' = \frac{\mu_0}{2\pi s} \frac{\lambda u}{\sqrt{1 - u^2/c^2}} \hat{\phi} \]

Which is what we derived from considering how the charge density & current changed due to length contraction & relativistic addition of velocities.

Now, if you carefully compare w/ the expressions from class, you'll notice that \( \epsilon_0 \mu_0 = \frac{1}{c^2} \). We'll explain why later. For now, can you show the following using the results above?

\[ \vec{E} \cdot \vec{E} - c^2 \vec{B} \cdot \vec{B} = \vec{E}' \cdot \vec{E}' - c^2 \vec{B}' \cdot \vec{B}' \]