▨ FIELDS & LORENTZ TRANSFORMATIONS

- In class we saw that observers in two different frames of reference may disagree on things like charge densities, currents, and fields. How does this work?

- First, let's remind ourselves how our spacetime description of an event changes between reference frames.

- Suppose I am in an "inertial" reference frame. That is, Newton's Laws accurately describe my observations about Forces, accelerations, etc.

- I will describe where things happen using a coordinate system w/ me @ its origin, and when they happen by reading a time off my watch.

- Meanwhile, you zip past me moving w/ constant velocity $\vec{u} = u\hat{u}$. We are briefly in the same place as you pass by, and we sync our watches @ that instant so they both read '0'.

- So as a result, we agree on the 'where' and 'when' of that event. I say that you passed me @ $\vec{r}=0$ & $t=0$ moving w/ velocity $\vec{u}=u\hat{u}$. You say that I passed you @ $\vec{r}\,'=0$ & $t'=0$ moving w/ velocity $-u\hat{u}$.

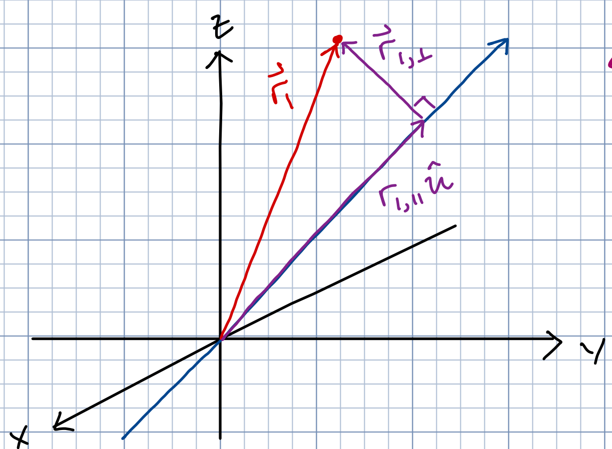
- Now suppose I notice some event that happens @ position $\vec{r}_1$ & time t_1. You will assign different spacetime coords to that event.

- First, let me orient my axes so that one of them is the dir. $\hat{u}$ I see you moving in.

- That is:

$$\vec{r}_1 = r_{1,\parallel}\hat{u} + \vec{r}_{1,\perp} \quad w/ \quad \vec{r}_{1,\perp}\cdot\hat{u} = 0$$

So if you were moving in what I called the $\hat{x}$ direction, then $r_{1,\parallel}$ would be x_1 & $\vec{r}_{1,\perp}$ would be $y_1\hat{y} + z_1\hat{z}$. But this lets us write our results more generally.



← I can write $\vec{r}_1$ as some distance $r_{1,\parallel}$ along the direction $\hat{u}$ of your velocity, and then a vector $\vec{r}_{1,\perp}$ in a plane that is $\perp$ to $\hat{u}$.

- In your frame of reference, you will describe this event differently. You'll agree w/ my $\vec{r}_{1,\perp}$ but not t_1 or $r_{1,\parallel}$:

$$t_1' = \gamma(u)\times\left(t_1 - \beta(u)\frac{1}{c}r_1\right) \qquad \beta(u) = \frac{u}{c}$$

$$r_{1,\parallel}' = \gamma(u)\times\left(r_{1,\parallel} - \beta(u)\,c\,t_1\right) \qquad \gamma(u) = \frac{1}{\sqrt{1-\beta(u)^2}}$$

$$\vec{r}_{1,\perp}' = \vec{r}_{1,\perp}$$

- This is the same Lorentz transformation you've seen before, written in a way that leaves the direction of $\vec{u}$ arbitrary. (Check that this looks like what you expect for $\vec{u} = u\hat{x}$!)

- Now, suppose I see a collection of charge & currents. I'll describe them in terms of a charge density ρ and a current density $\vec{J}$, which I'll write as:

$$\vec{J} = \underbrace{J_\parallel}_{\text{Current moving in same dir. as you from my P.o.V.}}\hat{u} + \underbrace{\vec{J}_\perp}_{\text{Current moving }\perp\text{ to your velocity, from my P.o.V.}} \quad w/ \quad \vec{J}_\perp\cdot\hat{u} = 0$$

- You will describe the same collection of charges & currents differently. That makes sense; something I say is @ rest will appear to be moving w/ vel. $-u\hat{u}$ in your frame of reference.

$$\rho' = \gamma(u) \times \left(\rho - \beta(u) \tfrac{1}{c} J_{\parallel}\right)$$
$$J_{\parallel}' = \gamma(u) \times \left(J_{\parallel} - \beta(u)\, c\, \rho\right)$$
$$\vec{J}_{\perp}' = \vec{J}_{\perp}$$

- This also applies to a line charge density λ & a current $\vec{I} = I_{\parallel}\hat{u} + \vec{I}_{\perp}$:

$$\lambda' = \gamma(u) \times \left(\lambda - \beta(u) \tfrac{1}{c} I_{\parallel}\right)$$
$$I_{\parallel}' = \gamma(u) \times \left(I_{\parallel} - \beta(u)\, c\, \lambda\right)$$
$$I_{\perp}' = I_{\perp}.$$

- So, suppose I see a line charge density λ consisting of a row of evenly spaced charges, but no current ($\vec{I} = 0$). You would interpret that as

$$\lambda' = \gamma(u)\,\lambda = \frac{\lambda}{\sqrt{1 - u^2/c^2}}$$

From your P.o.V., charges are moving w/ vel. $-u\hat{u}$. They appear to be spaced closer together b/c of <u>LENGTH CONTRACTION</u>.

$$I_{\parallel}' = -\gamma(u)\,\beta(u)\,c\lambda = -\frac{u\lambda}{\sqrt{1 - u^2/c^2}}$$

A line charge λ' moving w/ velocity $-u\hat{u}$ gives a current $-\lambda' u\hat{u}$.

$$\vec{I}_{\perp}' = 0$$

- Likewise, if I saw a line charge λ moving w/ vel. $u\hat{u}$ (so $\vec{I} = \lambda u \hat{u}$) you would perceive that as a stationary charge density:

$$\lambda' = \gamma(u) \times \left(\lambda - \beta(u) \tfrac{1}{c} \lambda u\right) = \frac{1}{\sqrt{1 - u^2/c^2}} \times \left(1 - \frac{u^2}{c^2}\right)\lambda = \sqrt{1 - u^2/c^2}\;\lambda$$
$$I_{\parallel}' = \gamma(u) \times \left(\lambda u - \beta(u)\, c\, \lambda\right) = 0$$
$$\vec{I}_{\perp}' = 0$$

- Since you & I tend to disagree on charge & current densities, we naturally disagree on the fields they produce.

- If I say that a collection of charges & currents produce an electric field $\vec{E}$ & magnetic field $\vec{B}$, then you see different fields $\vec{E}'$ & $\vec{B}'$.

- Let me write $\vec{E}$ & $\vec{B}$ as

$$\vec{E} = E_{\parallel}\hat{u} + \vec{E}_{\perp} \qquad \vec{B} = B_{\parallel}\hat{u} + \vec{B}_{\perp}$$

$E_{\parallel} = \vec{E}\cdot\hat{u}$ is the part of $\vec{E}$ in the direction of your vel.

$\vec{E}_{\perp}$ is the part of $\vec{E}$ perp. to your velocity: $\vec{E}_{\perp}\cdot\hat{u} = 0$

- Then the fields you see are

$$\vec{E}' = E_{\parallel}\hat{u} + \gamma(u)\vec{E}_{\perp} + \gamma(u)\,\vec{u}\times\vec{B}_{\perp}$$

$$\vec{B}' = B_{\parallel}\hat{u} + \gamma(u)\vec{B}_{\perp} - \gamma(u)\frac{1}{c^2}\,\vec{u}\times\vec{E}_{\perp}$$

- Notice that we <u>agree</u> on the components of the fields in the direction $\hat{u}$! It's the parts perp. to $\vec{u}$ that we dis-agree on.

- Let's be a bit more explicit. Imagine that we have two unit vectors $\hat{v}$ & $\hat{w}$ that are $\perp$ to $\hat{u}$ & to each other. Together, $\hat{u}$, $\hat{v}$, & $\hat{w}$ make up a right-handed coord. system w/

$$\hat{u}\times\hat{v} = \hat{w} \qquad \hat{v}\times\hat{w} = \hat{u} \qquad \hat{w}\times\hat{u} = \hat{v}$$

- The components of $\vec{E}_{\perp}$ & $\vec{B}_{\perp}$ can be written

$$\vec{E}_{\perp} = E_v\,\hat{v} + E_w\,\hat{w} \qquad \vec{B}_{\perp} = B_v\,\hat{v} + B_w\,\hat{w}$$

$$\hookrightarrow\ \vec{u}\times\vec{E}_{\perp} = -uE_w\,\hat{v} + uE_v\,\hat{w} \qquad \vec{u}\times\vec{B}_{\perp} = -uB_w\,\hat{v} + uB_v\,\hat{w}$$

- So if I observe fields

$$\vec{E} = \underbrace{E_u \hat{u}}_{E_{\parallel}} + \underbrace{E_v \hat{v} + E_w \hat{w}}_{\vec{E}_{\perp}} \qquad \vec{B} = \underbrace{B_u \hat{u}}_{B_{\parallel}} + \underbrace{B_v \hat{v} + B_w \hat{w}}_{\vec{B}_{\perp}}$$

Then the fields you measure are

$$E_u' = E_u \qquad E_v' = \gamma(u) \times (E_v - u B_w) \qquad E_w' = \gamma(u) \times (E_w + u B_v)$$

$$B_u' = B_u \qquad B_v' = \gamma(u) \times \left(B_v + \frac{u}{c^2} E_w\right) \qquad B_w' = \gamma(u) \times \left(B_w - \frac{u}{c^2} E_v\right)$$

- Does this agree w/ what we saw in class? We considered an electrically neutral wire carrying a current $I = \lambda v \hat{z}$. It produced no electric field, & a magnetic field $\vec{B} = (\mu_0 I / 2\pi s)\,\hat{\phi}$. In a frame of reference moving w/ $\vec{u} = u\hat{z}$, you'd see:

$$\hat{u} = \hat{z} \qquad \hat{v} = \hat{s} \qquad \hat{w} = \hat{\phi} \qquad \vec{E} = 0 \qquad \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{w} = \frac{\mu_0 \lambda v}{2\pi s} \hat{w}$$

$$\hookrightarrow \; E_u' = 0 \qquad E_s' = \gamma(u) \times \left(0 - u \frac{\mu_0 \lambda v}{2\pi s}\right) \qquad E_\phi' = 0$$

$$B_u' = 0 \qquad B_s' = 0 \qquad B_\phi' = \gamma(u) \times \left(\frac{\mu_0 \lambda v}{2\pi s} - 0\right)$$

$$\hookrightarrow \; \vec{E}' = -\frac{\mu_0}{2\pi s} \frac{\lambda u v}{\sqrt{1 - u^2/c^2}} \hat{s} \qquad \vec{B}' = \frac{\mu_0}{2\pi s} \frac{\lambda v}{\sqrt{1 - u^2/c^2}} \hat{\phi}$$

Which is what we derived from considering how the charge density & current changed due to length contraction & relativistic addition of velocities.

- Now, if you carefully compare w/ the expressions from class, you'll notice that $\varepsilon_0 \mu_0 = 1/c^2$. We'll explain why later. For now, can you show the following using the results above?

$$\vec{E} \cdot \vec{E} - c^2 \vec{B} \cdot \vec{B} = \vec{E}' \cdot \vec{E}' - c^2 \vec{B}' \cdot \vec{B}'$$