▨ LINE INTEGRALS

- A line integral follows a path P from point A to point B, and at every point along the way it adds an infinitesimal quantity (the integrand) to a running tally.
- The notation looks like this

$$\int_{P\ A}^{B} (\text{integrand})$$

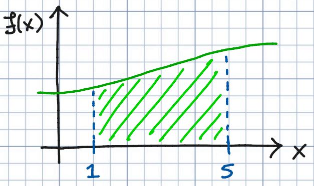
End point

The thing we are summing along the path.

Start point

Requires a path from A to B

- The simplest line integral is the definite integral you learned about in single-variable calculus.



$$\int_{1}^{5} dx\, f(x)$$

Start @ x=1 & end @ x=5, the path is along the x-axis & we add up the areas dx f(x) of an infinite # of infinitesimally narrow rectangles.

- But line integrals can involve more complicated paths, which don't have to be straight.
- We need to know how to get from one point on P to the next, nearby point. That's the infinitesimal displacement vector $d\vec{l}$

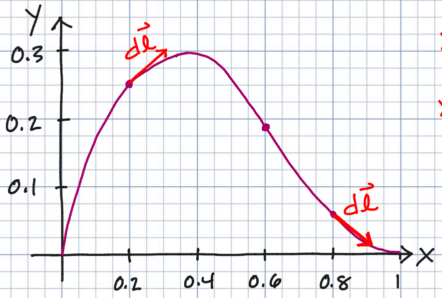
$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

The precise displacements dx, dy, & dz depend on the shape of the path. In the example above, the path is along the x-axis & $d\vec{l} = dx\,\hat{x}$.

- Suppose we wanted to integrate along a path in the x-y plane described by $y(x) = 2x^3 - 4x^2 + 2x$.

$$dy(x) = \frac{dy(x)}{dx}dx = 6x^2dx - 8x\,dx + 2dx$$
$$= (6x^2 - 8x + 2)\,dx$$

↳ $d\vec{l} = dx\,\hat{x} + (6x^2 - 8x + 2)dx\,\hat{y}$



$x = 0.2 \Rightarrow d\vec{l} = dx\,\hat{x} - 0.64dx\,\hat{y}$

$x = 0.8 \Rightarrow d\vec{l} = dx\,\hat{x} - 0.56dx\,\hat{y}$

↖ Notice how x & y have ← to change in the right proportion to keep you on the path.

- There are four different kinds of integrands we'll consider. They are built from $d\vec{l}$ or its magnitude $dl = |d\vec{l}|$, and either a scalar function h or a vector function $\vec{V}$.

- The integrand can be a scalar or a vector. If the integrand is a scalar then the result of the integral is a scalar. If the integrand is a vector, then the result of the integral is a vector. Nothing tricky here: adding up a bunch of little numbers gives you a number, and adding up a bunch of little vectors gives you a vector.

SCALAR INTEGRANDS: $dl\,h$ or $d\vec{l} \cdot \vec{V}$

VECTOR INTEGRANDS: $d\vec{l}\,h$ or $dl\,\vec{V}$

- So, given a path P and something to integrate, we follow these steps:

  (1) Describe the path P & infinitesimal displacement vector along the path.

  (2) Work out the integrand ($d\vec{l} \cdot \vec{V}$, $dl\, h$, etc) for points on P.

  (3) Evaluate the integral.

EXAMPLE: Evaluate the integral

$$\int_{P\ (0,0)}^{(1,-1)} dl\ x^2 y$$

along the path $y = x^2 - 2x$ in the x-y plane.

The integrand is the scalar quantity $dl\ x^2 y$, and the path is $y(x) = x^2 - 2x$ in the x-y plane.

$$y(x) = x^2 - 2x \rightarrow dy = (2x-2)\,dx$$

$$d\vec{l} = dx\,\hat{x} + (2x-2)\,dx\,\hat{y}$$

$$\Rightarrow dl = |d\vec{l}| = \left(dx^2 + (2x-2)^2 dx^2\right)^{1/2}$$

$$= dx\sqrt{4x^2 - 8x + 5}$$

Points on the path have $y = x^2 - 2x$, so the integrand is:

$$dl\ x^2 y = dx\sqrt{4x^2 - 8x + 5}\ \ x^2 \cdot (x^2 - 2x)$$

So the integral is

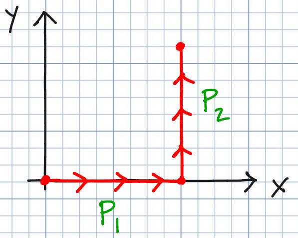
$$\int_{P\,(0,0)}^{(1,-1)} d\ell\, x^2 y = \int_0^1 dx \sqrt{4x^2-8x+5}\,\left(x^4-2x^3\right)$$

← Notice that we are integrating over <u>one</u> variable.

$$\vdots \quad \text{(complicated)}$$

$$= -0.34$$

<u>EXAMPLE</u>: Integrate $d\ell\,(xy+4x)$ from $(0,0)$ to $(2,2)$ along the path $(0,0) \to (2,0) \to (2,2)$.



$$\underbrace{\int_{P\,(0,0)}^{(2,2)} d\ell\,(xy+4x) = \int_{P_1\,(0,0)}^{(2,0)} d\ell\,(xy+4x) + \int_{P_2\,(2,0)}^{(2,2)} d\ell\,(xy+4x)}_{\text{Break the path into two parts.}}$$

PATH 1: $y=0$, $x: 0 \to 2$

$dy = 0$ since $y = \text{const.}$ $\Rightarrow$ $d\vec{\ell}_1 = dx\,\hat{x}$

$d\ell_1 = dx$

$$(xy+4x)\Big|_{y=0} = 4x$$

$$\to \int_{P_1\,(0,0)}^{(2,0)} d\ell\,(xy+4x) = \int_0^2 dx\; 4x = 2x^2\Big|_0^2 = 8$$

PATH 2: $x=2$, $y: 0 \to 2$ $\quad x = \text{const.}$ $\Rightarrow$ $dx = 0$

$d\vec{\ell}_2 = dy\,\hat{y}$ $\to$ $d\ell_2 = dy$

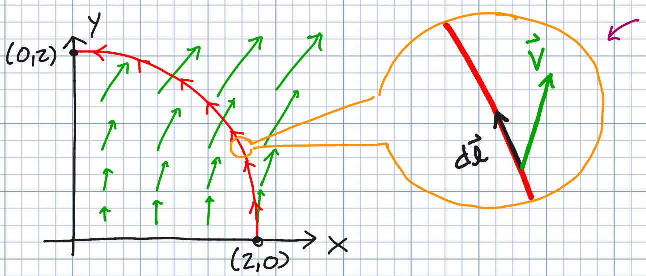
$$(xy+4x)\Big|_{x=2} = 2y+8$$

$$\to \int_{P_2\,(2,0)}^{(2,2)} d\ell\,(xy+4x) = \int_0^2 dy\,(2y+8) = (y^2+8y)\Big|_0^2 = 20$$

The total integral is the sum of the contributions from each part of the path:

$$\int_{P\,(0,0)}^{(2,2)} d\ell\,(xy+4x) = 8 + 20 = 28$$

<u>EXAMPLE</u>: For the vector $\vec{V} = xy\,\hat{x} + 2x^2\,\hat{y}$ in the x-y plane, integrate $d\vec{\ell}\cdot\vec{V}$ along a circular arc, in the CCW direction, from (2,0) to (0,2).



At each point on the path, determine the scalar quantity $d\vec{\ell}\cdot\vec{V}$ & add that to a running tally.

The path is a circular arc, so it's probably easiest to describe using polar coordinates. It has radius 2, so:

$$x(\phi) = 2\cos\phi \;\rightarrow\; dx = \frac{dx}{d\phi}\,d\phi = -2\sin\phi\,d\phi$$

$$y(\phi) = 2\sin\phi \;\rightarrow\; dy = \frac{dy}{d\phi}\,d\phi = 2\cos\phi\,d\phi$$

$$\Rightarrow d\vec{\ell} = -2\sin\phi\,d\phi\,\hat{x} + 2\cos\phi\,d\phi\,\hat{y}$$

At a point on the path, $\vec{V}$ is:

$$\vec{V} = xy\,\hat{x} + 2x^2\,\hat{y} = 4\cos\phi\sin\phi\,\hat{x} + 8\cos^2\phi\,\hat{y}$$

$$\hookrightarrow d\vec{\ell}\cdot\vec{V} = \left(-8\cos\phi\,\sin^2\phi + 16\cos^3\phi\right)d\phi$$

So the integral is:

Note: (2,0) is @ $\phi = 0$ & (0,2) is @ $\phi = \pi/2$.

$$\int_{P\ (2,0)}^{(0,2)} d\vec{l} \cdot \vec{V} = \int_0^{\pi/2} d\phi \left(-8\cos\phi \sin^2\phi + 16\cos^3\phi\right)$$

$\vdots$ (work out the integral)

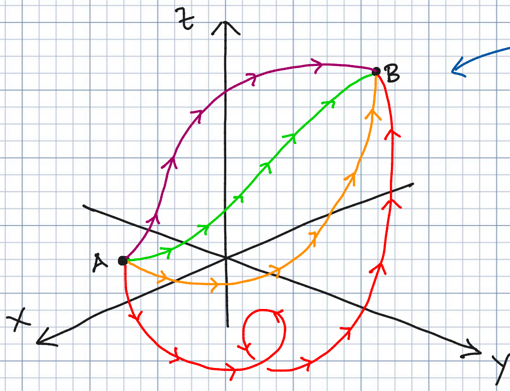
$$= 8$$

- Notice that a line integral always involves an integral over one variable (like the 1st & 3rd example) or a sum of integrals over single variables (like in the 2nd example).

- This is b/c a line integral is a way off adding up some quantity along a path, and we only need one number to specify where we are along a path or curve.

- If you are trying to evaluate a line integral and you end up with an integral over two or three variables, something went wrong!

- The result may or may not depend on the path from A to B!



These paths from A to B visit different points & curve around in different directions, so integrating the same function along them is likely to give different values. The exception is integrands of the form $d\vec{l} \cdot \vec{\nabla} h(x,y,z)$!

▨ SURFACE INTEGRALS

- A surface integral visits every point on a surface S, and adds an infinitesimal quantity to a running sum.
- The integrand in this case is an infinitesimal patch of area that we call 'da' times some quantity that may vary from point to point.
- Since we need two numbers to specify exactly where we are on a surface (latitude & longitude on a sphere, or x,y coords on a plane), a surface integral is always one or more double integrals over two variables.
- As w/ line integrals, the integrand can be a scalar or a vector.
- Our notation is:

$\int_S da\, h$	Visit every point on S, multiply da @ that point times the value of h @ that point, and add to the total. Result is a scalar.
$\int_S da\, \vec{V}$	Same as above, but adding up da @ each point times some vector $\vec{V}$. Result is a vector.
$\int_S da\, \hat{n}\cdot\vec{V}$	At each point, take the normal vector $\hat{n}$ (unit length, perpendicular to surface) & dot it w/ $\vec{V}$ @ that point. Multiply by da & add that to the total. Result is a scalar.

- If the surface is closed (has a definite inside & outside) we put an 'o' on the integral: $\oint$

- The area element da depends on the surface & the coordinates used to describe it.

- For a closed surface (like a sphere) we <u>always</u> think of $\hat{n}$ as pointing from inside to outside. For an open surface (like a disc) there are always two possibilities for the direction of $\hat{n}$. So you pick one & stick w/ it throughout the calculation.

- The combination $da\,\hat{n}$ is called the <u>Vector Area Element</u> and written $d\vec{a}$.

- Integrating $d\vec{a}\cdot\vec{V}$ over a surface gives the <u>FLUX</u> of $\vec{V}$ across or through the surface. The flux gives us an idea 'how much' $\vec{V}$ passes from one side of the surface to the other.

<u>EXAMPLE</u>: Integrate $h(x,y) = 3x^2y$ over the region $1 \le x \le 5$, $2 \le y \le 7$ in the x-y plane.

The surface is a rectangle in the x-y plane. To move from point-to-point on this surface we increment x by dx and y by dy, so we are 'tiling' it with little rectangles of area $da = dx\,dy$.

$$\int_S da\; h = \int_1^5 dx \int_2^7 dy\; 3x^2y = \int_1^5 dx \left(\frac{3}{2}x^2y^2 \Big|_2^7 \right)$$

$$= \int_1^5 dx\; \frac{135}{2}x^2 = \frac{135}{6}x^3\Big|_1^5 = 2790$$

EXAMPLE: Find the flux of $\vec{V} = 3\hat{x} + 2\hat{y} + 5\hat{z}$ through the hemisphere $x^2+y^2+z^2=4$ w/ $y \geq 0$.

This would be really complicated to work out in Cartesian coords, so we will use spherical polar coords:

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

The sphere has radius 2, so $r=2$ on the sphere. Normally $0 \leq \theta \leq \pi$ & $0 \leq \phi < 2\pi$ for a full sphere, but since we just want the half w/ $y \geq 0$ the coordinate ϕ takes the values $0 \leq \phi \leq \pi$. So in SPC the two coords that tell us where we are on this surface are $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi$.

From Math Methods we remember that the area element on the surface of a sphere is $r^2\sin\theta\, d\theta\, d\phi$. Since $r=2$, we have:

$$da = 4\sin\theta\, d\theta\, d\phi$$

$$da = h_\theta h_\phi\, d\theta\, d\phi$$
$$= r^2\sin\theta\, d\theta\, d\phi$$

To calculate the flux we need to find $\hat{n}\cdot\vec{V}$ on the hemisphere. We'll use $\hat{r}$ for the normal (since it's an open surface we could have used $-\hat{r}$). So

$$\hat{n}\cdot\vec{V} = 3\,\hat{r}\cdot\hat{x} + 2\,\hat{r}\cdot\hat{y} + 5\,\hat{r}\cdot\hat{z}$$

Remember? From Math Methods?

$$= 3\sin\theta\cos\phi + 2\sin\theta\sin\phi + 5\cos\theta$$

$$\hookrightarrow \int_S d\vec{a}\cdot\vec{V} = \int_0^\pi d\theta \int_0^\pi d\phi\; 4\sin\theta \times \left(3\sin\theta\cos\phi + 2\sin\theta\sin\phi + 5\cos\theta\right)$$

Integrate over ϕ

$$= \int_0^\pi d\theta\; 4\sin\theta \times \left(4\sin\theta + 5\pi\cos\theta\right)$$

Integrate over θ

$$= 8\pi$$

EXAMPLE: Integrate $h(x,y) = x^2+y^2$ over the surface of the oblate spheroid described by the equation

$$\frac{x^2}{2} + \frac{y^2}{2} + z^2 = 1 \leftarrow$$

It's a 'squashed' sphere, longer along the x & y axes.

This is much more complicated than the integrals we'll do in here b/c it involves some unusual coordinates. The analog of SPC for this shape are

$$x = \cosh\mu \cos\nu \cos\varphi$$
$$y = \sinh\mu \cos\nu \sin\varphi \qquad \text{w/ } \mu \geq 0,\ -\tfrac{\pi}{2} \leq \nu \leq \tfrac{\pi}{2},\ -\pi \leq \varphi < \pi$$
$$z = \sinh\mu \sin\nu$$

The surface we want corresponds to $\sinh\mu = 1$ in these coords, the same way $r=2$ in SPC corresponds to a sphere of radius 2. The scale factors for ν & φ are

$$h_\nu = \sqrt{\sinh^2\mu + \sin^2\nu} \qquad h_\varphi = \cosh\mu \cos\nu$$

So on our surface w/ $\sinh\mu = 1$ (& therefore $\cosh\mu = \sqrt{2}$) the area element is

$$da = h_\nu h_\varphi \, d\nu \, d\phi = \sqrt{2} \cos\nu \sqrt{1+\sin^2\nu}$$

The function we want to integrate over the surface is x^2+y^2. In these coords, that's

$$x^2+y^2 = \cosh^2\mu \cos^2\nu \cos^2\varphi + \sinh^2\mu \cos^2\nu \sin^2\varphi$$
$$= \cos^2\nu \times \left(2\cos^2\varphi + \sin^2\varphi\right)$$

So to summarize:

(1) Every point on the surface has a pair of coords (v, φ). To visit every point on the surface we need to consider all $-\pi/2 \leq v \leq \pi/2$ and $-\pi \leq \varphi \leq \pi$.

(2) The area element @ a point on this surface is $da = \sqrt{2}\,\cos v\,\sqrt{1+\sin^2 v}$.

(3) The function x^2+y^2 @ a point on this surface w/ coords (v, φ) is $\cos^2 v\,(2\cos^2\varphi + \sin^2\varphi)$.

Putting this together

$$\int_S da\, h = \int_{-\pi}^{\pi} d\varphi \int_{-\pi/2}^{\pi/2} dv\, \sqrt{2}\,\cos^3 v\,\sqrt{1+\sin^2 v} \times \left(2\cos^2\varphi + \sin^2\varphi\right)$$

$$\vdots \leftarrow \text{It's doable. That's not the point.}$$

$$= \frac{3\pi}{2} \times \left(2 + \frac{5}{\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)$$

The point is to show you how the set-up is more or less the same as the 1st example. Describe the surface, figure out da, work out what the integrand is for points on the surface, then put it all together.

- Again, remember that specifying a particular point on a surface requires two coords. So visiting all of them means that we always integrate over two variables in a surface integral.

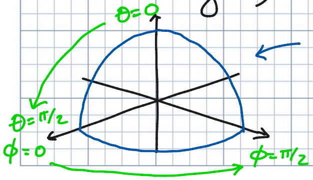
VOLUME INTEGRALS

- A volume integral visits every point inside a volume $\mathcal{V}$ and adds an infinitesimal quantity to a running sum.
- The integrand is an infinitesimal volume $d\tau$, multiplied by the value of some scalar or vector function.
- A volume is just some region of 3-D space. Since we need 3 coordinates to specify the location of a point inside $\mathcal{V}$, visiting every point means that a volume integral always involves integrating over 3 variables.
- The notation is:

$$\int_{\mathcal{V}} d\tau \, h \quad \text{or} \quad \int_{\mathcal{V}} d\tau \, \vec{V}$$

← Integrand can be a scalar or a vector.

EXAMPLE: Integrate the function $xy + z^2$ over the part of the inside of a unit sphere (centered at the origin) with $x, y, z \geq 0$.



← $\mathcal{V}$ is the inside of the unit sphere w/ $x, y, z \geq 0$. Let's use SPC: $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r \cos\theta$. Then $\mathcal{V}$ is $0 \leq r \leq 1$, $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq \pi/2$.

In SPC, the volume element is $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$.
And the function we're integrating is:

$$xy + z^2 = r^2 \sin^2\theta \sin\phi \cos\phi + r^2 \cos^2\theta$$

So the integral is:

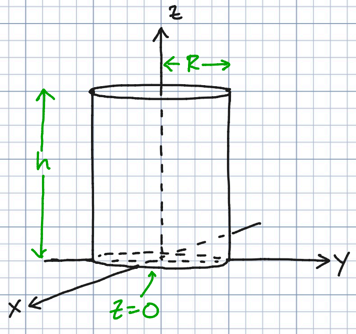
$$\int_V d\tau \, h = \underbrace{\int_0^1 dr \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi}_{\substack{\text{Add up contributions} \\ \text{from all pts with} \\ x,y,z \geqslant 0 \text{ inside the} \\ \text{unit sphere.}}} \underbrace{r^2 \sin\theta}_{} \times \overbrace{\left(r^2 \sin^2\theta \sin\phi \cos\phi + r^2 \cos^2\theta \right)}^{xy+z^2 \text{ in SPC}}$$

B/c $d\tau = dr d\theta d\phi \, r^2 \sin\theta$

$$= \frac{1}{15} + \frac{\pi}{30}$$

Again, the point here is setting up the integral, not evaluating it!

EXAMPLE: What is the average position of a point inside a cylinder w/ radius R & height h, with its base @ z=0 and its center along the z-axis?



The cylinder is easiest to describe in CPC:

$$x = s\cos\phi \qquad y = s\sin\phi \qquad z = z$$

Then V is the region $0 \leq s \leq R$, $0 \leq z \leq h$, $0 \leq \phi < 2\pi$. The volume element is $d\tau = s \, ds \, d\phi \, dz$.

But what is "average position"? A point inside the cylinder has position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ with $0 \leq \sqrt{x^2+y^2} \leq R$ and $0 \leq z \leq h$. How do we average that?

To get the average $\vec{r}$, which we will denote by $\langle \vec{r} \rangle$, we will add up $d\tau\,\vec{r}$ for every point inside $\mathcal{V}$ & then divide that by the sum of $d\tau$ for every point inside $\mathcal{V}$.

From $d\tau$!

$$\int_{\mathcal{V}} d\tau = \int_0^R ds \int_0^{2\pi} d\phi \int_0^h dz \; s \; = \pi R^2 h$$

$$\langle \vec{r} \rangle = \frac{\int_{\mathcal{V}} d\tau \, \vec{r}}{\int_{\mathcal{V}} d\tau}$$

$$\int_{\mathcal{V}} d\tau \, \vec{r} = \int_0^R ds \int_0^{2\pi} d\phi \int_0^h dz \; s \times \left(s\cos\phi \, \hat{x} + s\sin\phi \, \hat{y} + z \, \hat{z} \right)$$

$$= h \int_0^R ds \; s^2 \int_0^{2\pi} d\phi \left(\cos\phi \, \hat{x} + \sin\phi \, \hat{y} \right)$$

Integrating these over ϕ from 0 to 2π gives zero!

$$+ \int_0^R ds \; s \int_0^{2\pi} d\phi \int_0^h dz \; z \, \hat{z}$$

$$= 0\,\hat{x} + 0\,\hat{y} + \int_0^R ds \; s \int_0^{2\pi} d\phi \left(\tfrac{1}{2} z^2 \Big|_0^h \right) \hat{z}$$

$$= 0\,\hat{x} + 0\,\hat{y} + \int_0^R ds \; s \, \cancel{2}\pi \times \frac{1}{\cancel{2}} h^2 \, \hat{z}$$

$$= 0\,\hat{x} + 0\,\hat{y} + \frac{1}{2} \pi R^2 h^2 \, \hat{z}$$

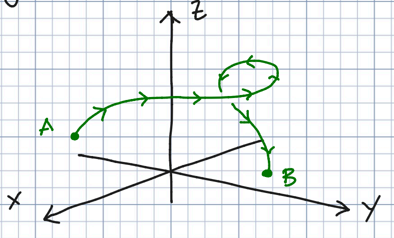
$$\hookrightarrow \langle \vec{r} \rangle = \frac{0\,\hat{x} + 0\,\hat{y} + \frac{1}{2}\pi R^2 h^2 \, \hat{z}}{\pi R^2 h} = 0\,\hat{x} + 0\,\hat{y} + \frac{h}{2} \, \hat{z}$$

$$\Rightarrow \langle \vec{r} \rangle = \frac{h}{2} \, \hat{z}$$

The average position of a point in the cylinder is right in the middle, as you'd expect!

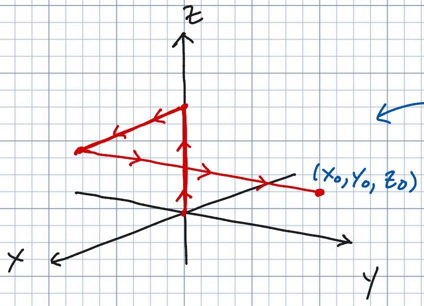
▨ THINGS TO REMEMBER...

- A line/path integral involves one or more integrals over a single variable. For instance:



Curve described by 3 functions $x(t), y(t), z(t)$ w/ pt A @ $x(0), y(0), z(0)$ and pt B @ $x(1), y(1), z(1)$. Integrate over t from 0 to 1:

$$\int_0^1 dt\,(\ldots)$$



Path is 3 straight lines from $(0,0,0) \to (0,0,z_0) \to (x_0,0,z_0) \to (x_0,y_0,z_0)$.

$$P\int_{(0,0,0)}^{(x_0,y_0,z_0)} d\vec{l}\cdot\vec{V}(x,y,z) = \int_0^{z_0} dz\,\hat{z}\cdot\vec{V}(0,0,z) + \int_0^{x_0} dx\,\hat{x}\cdot\vec{V}(x,0,z_0)$$

$$+\int_0^{y_0} dy\,\hat{y}\cdot\vec{V}(x_0,y,z_0)$$

- A surface integral involves one or more integrals over two variables. (More than one integral if you have to break the surface up into multiple parts, each w/ its own description.)

- A volume integral involves integrating over 3 variables.
- Be careful & go step-by-step when setting up the integral. Given a curve, surface, or volume, figure out how to describe it mathematically. Then work out $d\vec{l}$ or $d\vec{a}$ or $d\tau$, and then work out the rest of the integrand @ the relevant points. Put it all together once you have each of these components!
- The fundamental theorems may let you avoid doing an integral, or may let you evaluate a different kind of integral:

$$_P\int_A^B d\vec{l}\cdot\vec{\nabla}h = h(B)-h(A) \leftarrow$$

Same for all paths from A to B!

$$\int_V d\tau\,\vec{\nabla}\cdot\vec{V} = \oint_S da\,\hat{n}\cdot\vec{V}$$

$$\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{V}) = \oint_P d\vec{l}\cdot\vec{V}$$