Legendre Polynomials: What They Are, How To Use Them

- Legendre's equation shows up all over the place in physics:

$$(1-x^2)y''(x) - 2x\,y'(x) + \ell(\ell+1)\,y(x) = 0 \qquad \ell \in \mathbb{Z},\ \ell \geq 0$$

- Why does it show up? Because the Laplacian in SPC has a part that looks like

$$\nabla^2 \Psi = \ldots + \frac{1}{r^2}\underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Psi}{\partial\theta}\right)}_{= \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial\Psi}{\partial x}\right) \text{ if } x=\cos\theta} + \ldots$$

A powerful technique for solving PDEs (Separation of variables) will have us look for sol'ns where this part of $\nabla^2\Psi$ equals a constant $-\ell(\ell+1)$, leading to Legendre!

- If we look for <u>series</u> <u>solutions</u> we find two. Since $\ell \geq 0$ is an integer, one of the solutions ($P_\ell(x)$) is a FINITE series. Since it only has a finite # of terms, it is finite $\forall\, x$. The other sol'n ($Q_\ell(x)$) is an infinite series & it converges only for $-1<x<1$.

$$y(x) = a\,P_\ell(x) + b\,Q_\ell(x)$$

} General Solution

- For $-1<x<1$ this is the most general sol'n of Legendre's equation.

- Often, we're studying some physical system and expect
  - (i) Whatever quantity $y(x)$ represents is well-behaved, something physical that is finite $\forall x$.
  - (ii) We want to know $y(x)$ for $-1 \le x \le 1$, including the points $x = \pm 1$ where Q_ℓ diverges. (For instance, b/c the variable x is related to θ in SPC by $x = \cos\theta$, so $x = \pm 1$ is $\theta = 0$ or $\theta = \pi$.)
- In that case, requiring $y(x)$ finite @ $x = \pm 1$ forces us to drop $Q_\ell(x)$ & focus on the $P_\ell(x)$ sol'ns.
- The $P_\ell(x)$ for the 1st several values of ℓ are:

ℓ	$P_\ell(x)$
0	1
1	x
2	$\frac{3}{2}x^2 - \frac{1}{2}$
3	$\frac{5}{2}x^3 - \frac{3}{2}x$
4	$\frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$
5	$\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$
6	$\frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16}$
7	$\frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x$

- For any value of ℓ we can find $P_\ell(x)$ using RODRIGUES'S FORMULA.

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}\left((x^2-1)^\ell\right)$$

But this is a pretty cumbersome way to work them out – it's easier to look them up!

- The $P_\ell(x)$ are polynomials of order x^ℓ. We typically use them on $-1 \le x \le 1$. If ℓ is even then $P_\ell(x)$ is even on $-1 \le x \le 1$, and if ℓ is odd then $P_\ell(x)$ is odd on $-1 \le x \le 1$. They are normalized so that $P_\ell(1) = 1 \;\; \forall \, \ell$. Then $P_\ell(-1) = 1$ if ℓ is even, and $P_\ell(-1) = -1$ if ℓ is odd.

- Legendre polynomials of the 1st kind have a GENERATING FUNCTION $G(x,h)$. That's a function that gives the $P_\ell(x)$ as coefficients of h^ℓ in a Maclaurin series expansion:

$$G(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} \qquad -1 \le x \le 1, \quad -1 < h < 1$$

$$= \sum_{\ell=0}^{\infty} h^\ell P_\ell(x)$$

- Working w/ the Maclaurin series expansion of $G(x,h)$ & its derivatives, we get lots of useful identities satisfied by the $P_\ell(x)$.

- These identities, which relate $P_\ell(x)$ & its derivatives for different values of ℓ, are called "Recurrence relations." Here are a few useful ones:

(1) $\ell P_\ell(x) = (2\ell-1)\, x\, P_{\ell-1}(x) - (\ell-1) P_{\ell-2}(x)$

(2) $x \frac{d}{dx} P_\ell(x) = \ell P_\ell(x) + \frac{d}{dx} P_{\ell-1}(x)$

(3) $\frac{d}{dx} P_\ell(x) - x \frac{d}{dx} P_{\ell-1}(x) = \ell P_{\ell-1}(x)$

(4) $(1-x^2) \frac{d}{dx} P_\ell(x) = \ell P_{\ell-1}(x) - x\, \ell P_\ell(x)$

(5) $(2\ell+1) P_\ell(x) = \frac{d}{dx} P_{\ell+1}(x) - \frac{d}{dx} P_{\ell-1}(x)$

(6) $(1-x^2) \frac{d}{dx} P_{\ell-1}(x) = \ell\, x\, P_{\ell-1}(x) - \ell P_\ell(x)$

- For our purposes, the most important application of these sorts of identities is working out the integral of $P_\ell(x) P_k(x)$. Combining various identities, we find

$$\frac{d}{dx}\left((1-x^2)\left(P_\ell(x) \frac{d}{dx} P_k(x) - P_k(x) \frac{d}{dx} P_\ell(x)\right)\right)$$
$$+ \left(\ell(\ell+1) - k(k+1)\right) P_\ell(x) P_k(x) = 0$$

- Now suppose $\ell \neq k$. Then:

$$-\underbrace{\left(\ell(\ell+1)-k(k+1)\right)}_{\substack{\neq 0 \text{ if } \ell \neq k \\ (\ell, k \in \mathbb{Z},\ \geq 0!)}} \int_{-1}^{1} dx\, P_\ell(x) P_k(x) = \int_{-1}^{1} dx \frac{d}{dx}\left((1-x^2)\left(P_\ell(x) P_k'(x) - P_k(x) P_\ell'(x)\right)\right)$$

$$= \underbrace{(1-x^2)}_{=0 \,@\, x=\pm 1} \underbrace{\left(P_\ell(x) P_k'(x) - P_k(x) P_\ell'(x)\right)}_{\text{FINITE } @\ x = \pm 1} \Bigg|_{-1}^{1}$$

- So if $\ell \neq k$, the integral of $P_\ell(x)\,P_k(x)$ from -1 to 1 is <u>ZERO</u>.

$$\int_{-1}^{1} dx\, P_\ell(x)\, P_k(x) = 0 \qquad \ell \neq k$$

- What if $\ell = k$? Using identity (2) on the previous page:

$$\int_{-1}^{1} dx\, P_\ell(x) P_\ell(x) = \int_{-1}^{1} dx\, P_\ell(x) \times \frac{1}{\ell}\left(x P_\ell'(x) - P_{\ell-1}'(x) \right)$$

$$= \frac{1}{\ell} \int_{-1}^{1} dx\, \underbrace{x P_\ell(x) P_\ell'(x)}_{\frac{1}{2} x \frac{d}{dx}\left(P_\ell(x)^2\right) = \frac{d}{dx}\left(\frac{x}{2} P_\ell^2\right) - \frac{1}{2} P_\ell(x)^2} - \frac{1}{\ell} \int_{-1}^{1} dx\, P_\ell(x) P_{\ell-1}'(x)$$

$$= \frac{1}{2\ell} \int_{-1}^{1} dx\, \frac{d}{dx}\left(x P_\ell(x)^2 \right) - \frac{1}{2\ell} \int_{-1}^{1} dx\, P_\ell(x)^2$$
$$- \frac{1}{\ell} \int_{-1}^{1} dx\, P_\ell(x) P_{\ell-1}'(x)$$

$$\hookrightarrow \underbrace{\left(1 + \frac{1}{2\ell}\right)}_{\frac{2\ell+1}{2\ell}} \int_{-1}^{1} dx\, P_\ell(x)^2 = \frac{1}{2\ell}\left(\underbrace{x P_\ell(x)^2 \Big|_{-1}^{1}}_{1 \cdot \overset{1}{\cancel{P_\ell(1)}}^2 - (-1)\overset{1}{\cancel{P_\ell(-1)}}^2 \; = 2} \right) - \frac{1}{\ell} \int_{-1}^{1} dx\, P_\ell(x) P_{\ell-1}'(x)$$

$$\hookrightarrow \int_{-1}^{1} dx\, P_\ell(x)^2 = \frac{2}{2\ell+1} - \frac{2}{2\ell+1} \int_{-1}^{1} dx\, P_\ell(x) P_{\ell-1}'(x)$$

- This is almost what we want! IBP in:

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell-1}'(x) = \int_{-1}^{1} dx\, \frac{d}{dx}\left(P_\ell(x) P_{\ell-1}(x) \right) - \int_{-1}^{1} dx\, P_{\ell-1}(x) P_\ell'(x)$$

$$\hookrightarrow \int_{-1}^{1} dx\, P_\ell(x)^2 = \frac{2}{2\ell+1} - \frac{2}{2\ell+1}\left(\underbrace{P_\ell(x) P_{\ell-1}(x)\Big|_{-1}^{1}}_{\substack{1-(-1)^\ell(-1)^{\ell-1} \\ =2}} - \int_{-1}^{1} dx\, P_{\ell-1}(x)\, P_\ell'(x) \right)$$

$$= \frac{2}{2\ell+1} - \frac{2}{2\ell+1}\left(2 - \int_{-1}^{1} dx\, P_{\ell-1}(x) \underbrace{P_\ell'(x)}_{\substack{\ell P_{\ell-1}(x) + x P_{\ell-1}'(x) \\ \text{by identity (3)}}} \right)$$

$$= \frac{2}{2\ell+1} - \frac{2}{2\ell+1}\left(2 - \ell \int_{-1}^{1} dx\, P_{\ell-1}(x)^2 - \int_{-1}^{1} dx\, x P_{\ell-1} \underbrace{P_{\ell-1}'}_{\text{IBP AGAIN!}} \right)$$

$$= \frac{2}{2\ell+1} - \frac{2}{2\ell+1}\left(\underbrace{2 - 1}_{1} - \left(\ell - \tfrac{1}{2}\right) \int_{-1}^{1} dx\, P_{\ell-1}(x)^2 \right)$$

- So after all those IBP & identities, we have

$$\int_{-1}^{1} dx\, P_\ell(x)^2 = \frac{2}{2\ell+1} - \frac{2}{2\ell+1}\left(1 - \frac{2(\ell-1)+1}{2} \int_{-1}^{1} dx\, P_{\ell-1}(x)^2 \right)$$

Or, after a little algebra....

$$\frac{2\ell+1}{2} \int_{-1}^{1} dx\, P_\ell(x)^2 = \frac{2(\ell-1)+1}{2} \int_{-1}^{1} dx\, P_{\ell-1}(x)^2$$

- Now, $P_0(x) = 1$, so we can solve this recursively to find the integral!

$$I_\ell \equiv \int_{-1}^{1} dx\, P_\ell(x)^2 \quad \Rightarrow \quad \frac{2\ell+1}{2} I_\ell = \frac{2(\ell-1)+1}{2} I_{\ell-1}$$

$$\Rightarrow I_\ell = \frac{2(\ell-1)+1}{2\ell+1} I_{\ell-1}$$

$$I_\ell = \frac{2(\ell-1)+1}{2\ell+1} I_{\ell-1}$$

$$= \frac{\cancel{2(\ell-1)+1}}{2\ell+1} \times \frac{2(\ell-2)+1}{\cancel{2(\ell-1)+1}} I_{\ell-2}$$

$$= \dots$$

$$= \frac{2\cdot 0+1}{2\ell+1} I_0 = \frac{1}{2\ell+1} \overbrace{\int_{-1}^{1} dx}^{x\big|_{-1}^{1} = 1-(-1) = 2} \underbrace{P_0(x)^2}_{1} = \frac{2}{2\ell+1}$$

- So we get:

$$\int_{-1}^{1} dx\, P_\ell(x) P_k(x) = \begin{cases} 0 & \text{if } \ell \neq k \\ \frac{2}{2\ell+1} & \text{if } \ell = k \end{cases}$$

- Why did we work so hard for this result? B/c now we know that the $P_\ell(x)$ are a family of ORTHOGONAL FUNCTIONS on the interval $-1 \leq x \leq 1$! We can use them to build a series representation of a function $f(x)$ on $-1 \leq x \leq 1$, just like we did w/ Fourier series!

- Let $f(x)$ be a well-behaved (satisfies the Dirichlet conditions) function on $-1 \leq x \leq 1$. Then

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)$$

$$\text{w/} \quad a_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} dx\, f(x) P_\ell(x)$$

- Proof of a_ℓ:

$$f(x) = \sum_{k=0}^{\infty} c_k P_k(x)$$

$\rightarrow$ Mult. both sides by $P_\ell(x)$ & integrate

$$\hookrightarrow \int_{-1}^{1} dx\, P_\ell(x) f(x) = \sum_{k=0}^{\infty} c_k \int_{-1}^{1} dx\, P_\ell(x) P_k(x)$$

$$= c_0 \underbrace{\int_{-1}^{1} dx\, P_\ell(x) P_0(x)}_{0} + \dots + c_{\ell-1} \underbrace{\int_{-1}^{1} dx\, P_\ell(x) P_{\ell-1}(x)}_{0}$$

$$+ c_\ell \int_{-1}^{1} dx\, P_\ell(x)^2 + c_{\ell+1} \underbrace{\int_{-1}^{1} dx\, P_\ell(x) P_{\ell+1}(x)}_{0} + \dots$$

$$= c_\ell \cdot \frac{2}{2\ell+1}$$

$$\Rightarrow c_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} dx\, f(x) P_\ell(x)$$

- EXAMPLE: $f(x) = 6x^3 + 2x^2$

$$c_0 = \frac{1}{2} \int_{-1}^{1} dx\, f(x) \cancelto{1}{P_0(x)} = \frac{1}{2}\left(6\frac{x^4}{4} + 2\frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{1}{2}\cdot\left(0 + \frac{2}{3}\cdot 2\right) = \frac{2}{3}$$

$$c_1 = \frac{3}{2} \int_{-1}^{1} dx\, (6x^3 + 2x^2)\cdot x = \frac{3}{2}\left(6\cdot\frac{x^5}{5} + 2\cdot\frac{x^4}{4}\right)\Big|_{-1}^{1} = \frac{3}{\cancel{2}}\cdot\frac{6}{5}\cancel{2} = \frac{18}{5}$$

$$c_2 = \frac{5}{2} \int_{-1}^{1} dx\, (6x^3 + 2x^2)\left(\frac{3}{2}x^2 - \frac{1}{2}\right) = \frac{5}{2}\left(\frac{3}{2}x^6 + \frac{3}{5}x^5 - \frac{3}{4}x^4 - \frac{1}{3}x^3\right)\Big|_{-1}^{1} = \frac{4}{3}$$

$$c_3 = \frac{7}{2} \int_{-1}^{1} dx\, (6x^3 + 2x^2)\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) = \frac{7}{2}\left(\frac{15}{7}x^7 + \frac{5}{6}x^6 - \frac{9}{5}x^5 - \frac{3}{4}x^4\right)\Big|_{-1}^{1} = \frac{12}{5}$$

$$c_4 = \frac{9}{2} \int_{-1}^{1} dx\, (6x^3 + 2x^2) P_4(x) = 0$$

$$\int_{-1}^{1} dx\, x^k P_\ell(x) = 0 \quad \forall\ \ell > k$$

$$\Rightarrow f(x) = \frac{12}{5} P_3(x) + \frac{4}{3} P_2(x) + \frac{18}{5} P_1(x) + \frac{2}{3} P_0(x)$$

- Two important points for this last example.

- First, an important corollary of our result for integrals of $P_\ell(x) P_k(x)$. Since $P_k(x)$ is a polynomial of order x^k, we get:

$$\int_{-1}^{1} dx \, P_\ell(x) \, x^k = 0 \quad \forall \ \ell > k$$

This means that if $f(x)$ is a polynomial of order k, then all the coefficients c_ℓ w/ $\ell > k$ in its Legendre series representation are ZERO. In the previous example, $c_\ell = 0 \quad \forall \ \ell \geq 4$.

- Second, if $f(x)$ is a polynomial we can also find the c_ℓ algebraically!

$$6x^3 + 2x^2 = c_3 P_3(x) + c_2 P_2(x) + c_1 P_1(x) + c_0 P_0(x)$$

$$= \frac{5}{2} c_3 x^3 - \frac{3}{2} c_3 x + \frac{3}{2} c_2 x^2 - \frac{1}{2} c_2 + c_1 x + c_0$$

$$6x^3 + 2x^2 = \frac{5}{2} c_3 x^3 + \frac{3}{2} c_2 x^2 + \left(c_1 - \frac{3}{2} c_3\right) x + c_0 - \frac{1}{2} c_2$$

$$\Rightarrow \quad 6 = \frac{5}{2} c_3 \qquad 2 = \frac{3}{2} c_2 \qquad 0 = c_1 - \frac{3}{2} c_3 \qquad 0 = c_0 - \frac{1}{2} c_2$$

$$c_3 = \frac{12}{5} \qquad c_2 = \frac{4}{3} \qquad c_1 = \frac{3}{2} c_3 = \frac{18}{5} \qquad c_0 = \frac{1}{2} c_2 = \frac{2}{3}$$

- This only works if $f(x)$ is a polynomial in x!

- EXAMPLE: $f(x) = \sin(\pi x)$ ← Not a polynomial, get an infinite series!

$$C_0 = \frac{1}{2}\int_{-1}^{1} dx\, \sin(\pi x)\cdot 1 = 0$$

$$C_1 = \frac{3}{2}\int_{-1}^{1} dx\, \sin(\pi x)\cdot x = \frac{3}{2}\left(\frac{1}{\pi^2}\sin(\pi x) - \frac{x}{\pi}\cos(\pi x)\right)\Bigg|_{-1}^{1} = \frac{3}{\pi}$$

$$C_2 = \frac{5}{2}\int_{-1}^{1} dx\, \underbrace{\sin(\pi x)}_{\substack{\text{Odd on}\\ -1\le x\le 1}}\ \underbrace{P_2(x)}_{\substack{\text{Even on}\\ -1\le x\le 1}} = 0$$

← All $C_{2k} = 0$ b/c $P_{2k}(x)$ are even on $-1 \le x \le 1$

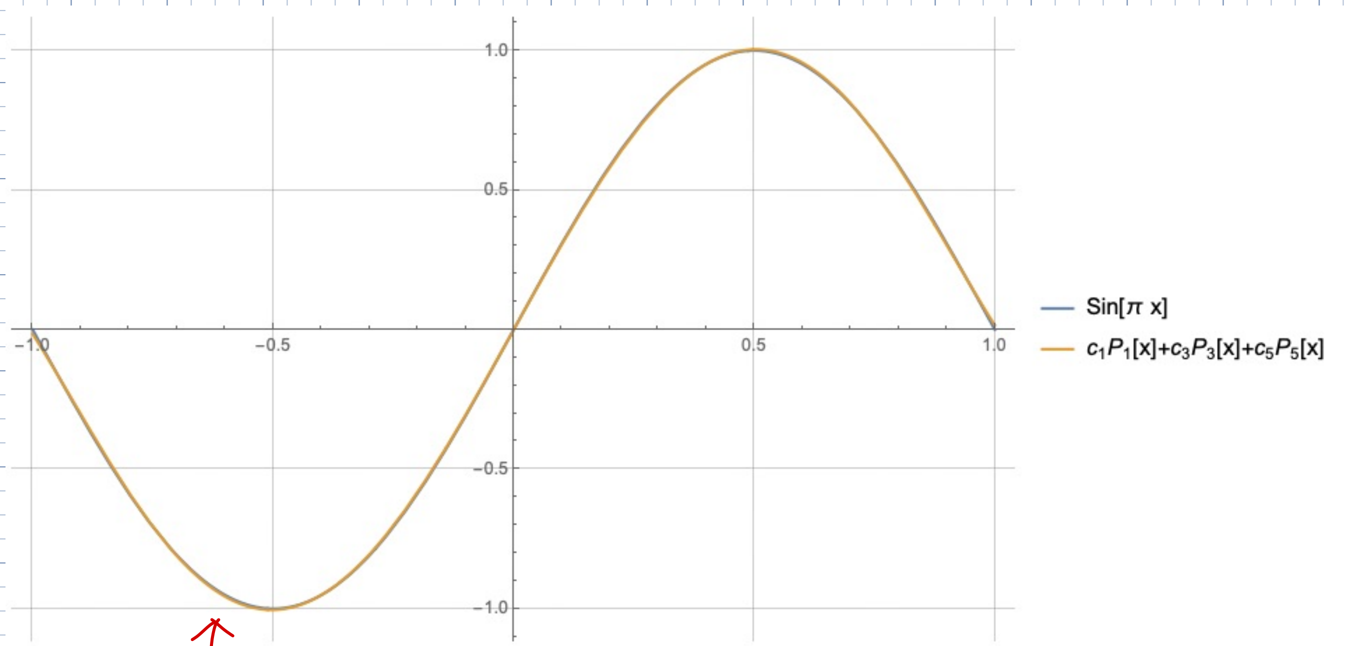
$$C_3 = \frac{7}{2}\int_{-1}^{1} dx\, \sin(\pi x)\, P_3(x) = \frac{7}{\pi} - \frac{105}{\pi^3}$$

← VIA MATHEMATICA!

$$C_5 = \frac{11}{2}\int_{-1}^{1} dx\, \sin(\pi x)\, P_5(x) = \frac{11}{\pi^5}\left(\pi^4 - 105\pi^2 + 945\right)$$

$\vdots$

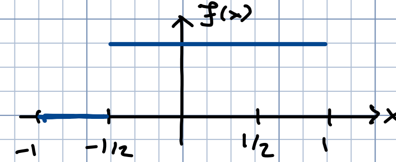
$$\sin(\pi x) = \frac{3}{\pi}P_1(x) + \frac{7}{\pi^3}(\pi^2 - 15)P_3(x) + \frac{11}{\pi^5}(\pi^4 - 105\pi^2 + 945)P_5(x) + \underbrace{\cdots}_{\substack{\text{An } \infty \text{ number of}\\ \text{odd terms.}}}$$



Just the 1st 3 terms ($\ell = 1,3,5$) give a very good approximation!

- Remember, if a function is defined piecewise we have to break the integral into different intervals.

- EXAMPLE: $f(x) = \begin{cases} 0, & -1 \leq x < -1/2 \\ 1, & -1/2 \leq x \leq 1 \end{cases}$



$$\underbrace{c_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} dx\, f(x)\, P_\ell(x)}_{\int_{-1}^{-1/2} dx \cdot 0 \cdot P_\ell(x) \; + \; \int_{-1/2}^{1} dx \cdot 1 \cdot P_\ell(x)}$$

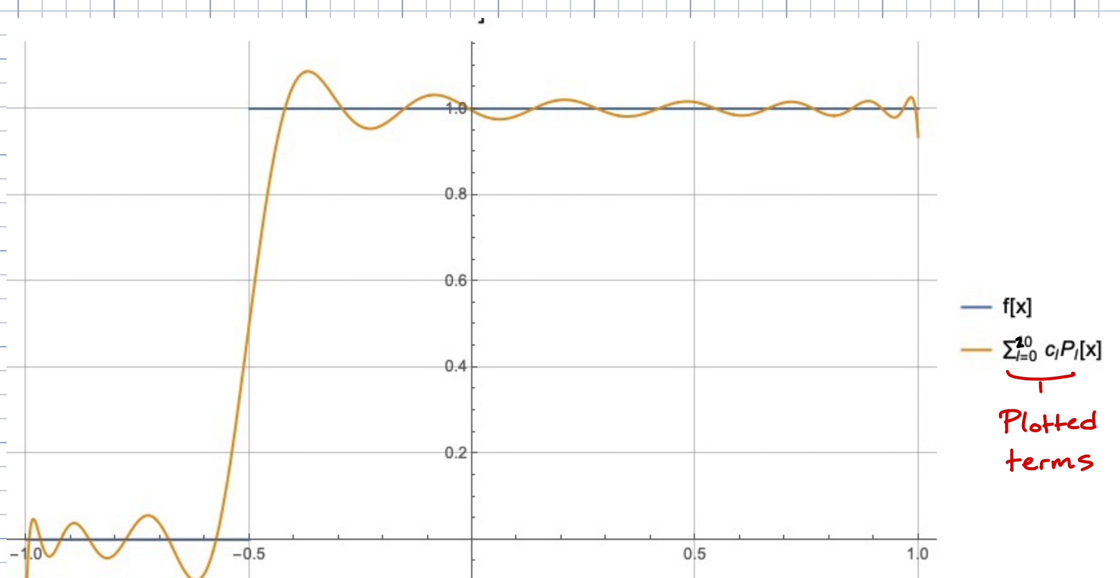
$$= \frac{2\ell+1}{2} \int_{-1/2}^{1} dx\; 1 \cdot P_\ell(x)$$

$$c_0 = \frac{1}{2} \cdot \int_{-1/2}^{1} dx \cdot 1 = \frac{x}{2}\Big|_{-1/2}^{1} = \frac{3}{4} \qquad c_1 = \frac{3}{2} \int_{-1/2}^{1} dx\; x = \frac{3x^2}{4}\Big|_{-1/2}^{1} = \frac{3}{4} - \frac{3}{16} = \frac{9}{16}$$

$$c_2 = \frac{5}{2} \int_{-1/2}^{1} dx \cdot \left(\frac{3}{2}x^2 - \frac{1}{2}\right) = \frac{5}{2}\left(\frac{1}{2}x^3 - \frac{1}{2}x\right)\Big|_{-1/2}^{1} = \frac{5}{2}\left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} - \frac{1}{2}\left(-\frac{1}{8}\right) + \frac{1}{2} \cdot \left(-\frac{1}{2}\right)\right) = -\frac{15}{32}$$

$$c_3 = \frac{7}{2} \int_{-1/2}^{1} dx \left(\frac{5}{2}x^3 - \frac{3}{2}x\right) = \frac{7}{2}\left(\frac{5}{8}x^4 - \frac{3}{4}x^2\right)\Big|_{-1/2}^{1} = \frac{21}{256} \quad \text{etc.}$$

$$\hookrightarrow f(x) = \frac{3}{4} + \frac{9}{16} P_1(x) - \frac{15}{32} P_2(x) + \frac{21}{256} P_3(x) + \frac{135}{512} P_4(x) + \ldots$$



Plotted 1st 20 terms

Notice Gibbs phenomenon @ $x = -1/2$ where $f(x)$ is discontinuous

- Legendre series representations will be a very important tool when we solve eqns like $\nabla^2 \Psi(r,\theta,\phi) = 0$.