

A Few Useful Integrals

Here is an integral that we'll see in class and on the homework:

$$\int dx \frac{1}{(x^2 + \beta^2)\sqrt{x^2 + \alpha^2}} = \frac{1}{\beta\sqrt{\alpha^2 - \beta^2}} \tan^{-1} \left(\frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \frac{x}{\sqrt{\alpha^2 + x^2}} \right) \quad (1)$$

How do we perform this integral? One way is to use a trig substitution, though it is a little hard to see where it comes from. Consider a right triangle where the side opposite an angle θ is $x\sqrt{\alpha^2 - \beta^2}$, and the side adjacent to the angle is $\beta\sqrt{x^2 + \alpha^2}$. (Notice that we definitely need $\alpha^2 > \beta^2$ so that the length of the opposite side is a \mathbb{R} number!) Then the hypotenuse has length $\alpha\sqrt{x^2 + \beta^2}$. Now look at the standard trig functions: $\sin \theta$, $\cos \theta$, and $\tan \theta$. For our right triangle we have

$$\sin \theta = \frac{x\sqrt{\alpha^2 - \beta^2}}{\alpha\sqrt{x^2 + \beta^2}} \quad (2)$$

$$\cos \theta = \frac{\beta\sqrt{x^2 + \alpha^2}}{\alpha\sqrt{x^2 + \beta^2}} \quad (3)$$

$$\tan \theta = \frac{x\sqrt{\alpha^2 - \beta^2}}{\beta\sqrt{x^2 + \alpha^2}}. \quad (4)$$

What does this do for us? Well, let's see what happens when we try to write our integral over x as an integral over θ . To convert the integral we need to know how dx and $d\theta$ are related. This isn't too hard – we can use the formula for $\sin \theta$ or $\cos \theta$ to figure it out. Let's use $\sin \theta$. Taking the derivative of both sides of (2) with respect to x , we wind up with (you should fill in the details here)

$$d\theta = \frac{\beta\sqrt{\alpha^2 - \beta^2}}{(x^2 + \beta^2)\sqrt{x^2 + \alpha^2}} dx \quad (5)$$

$$\Rightarrow dx \frac{1}{(x^2 + \beta^2)\sqrt{x^2 + \alpha^2}} = d\theta \frac{1}{\beta\sqrt{\alpha^2 - \beta^2}} \quad (6)$$

That's great! It leaves us with a very simple integral

$$\int dx \frac{1}{(x^2 + \beta^2)\sqrt{x^2 + \alpha^2}} = \frac{1}{\beta\sqrt{\alpha^2 - \beta^2}} \int d\theta \quad (7)$$

$$= \frac{1}{\beta\sqrt{\alpha^2 - \beta^2}} \theta \quad (8)$$

Now we just express θ in terms of our original variable x using any one of (2), (3), or (4). For instance

$$\frac{1}{\beta\sqrt{\alpha^2 - \beta^2}} \theta = \frac{1}{\beta\sqrt{\alpha^2 - \beta^2}} \tan^{-1} \left(\frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \frac{x}{\sqrt{\alpha^2 + x^2}} \right), \quad (9)$$

which is the result (1) we were looking for. When you encounter this integral on the homework you should have $\alpha^2 > \beta^2$, so that you get a \mathbb{R} number.

Another integral you will encounter on homework assignments is

$$\int d\theta \frac{\sin \theta}{(a + b \cos \theta)^n} \quad (10)$$

where n is a number, and a and b don't depend on θ .¹ This one is much easier to evaluate than the first integral; we just need a simple substitution to turn it into an elementary integral. First let's make a change of variables to simplify our denominator a bit.

$$u = a + b \cos \theta \quad du = -b \sin \theta d\theta . \quad (11)$$

This works because the integrand of (10) has the factor of $\sin \theta$ in its numerator. In the new variable the integral becomes

$$\int d\theta \frac{\sin \theta}{(a + b \cos \theta)^n} = -\frac{1}{b} \int du \frac{1}{u^n} . \quad (12)$$

I think you can handle it from here! It is tempting to work out the result for arbitrary n , and then set n to whatever value appears in the integral that shows up in the homework. But you should be careful: the answer you obtain for $n = 1$ is qualitatively different than the answer for $n \neq 1$.

Finally, an integral similar to (10) that shows up quite a bit is

$$\int d\theta \frac{\sin \theta \cos \theta}{(a + b \cos \theta)^n} \quad (13)$$

The trick we used for the last integral works here, too. With the change of variables (11) we get

$$\int d\theta \frac{\sin \theta \cos \theta}{(a + b \cos \theta)^n} = -\frac{1}{b^2} \int du \frac{1}{u^n} (u - a) \quad (14)$$

where we also used (11) to rewrite the extra factor of $\cos \theta$ as $(u - a)/b$. We arrive at

$$\int d\theta \frac{\sin \theta \cos \theta}{(a + b \cos \theta)^n} = \frac{a}{b^2} \int du \frac{1}{u^n} - \frac{1}{b^2} \int du \frac{1}{u^{n-1}} . \quad (15)$$

This time we need to do two integrals, but they are both elementary. Keep in mind, though, that if $n = 1$ or $n = 2$ you will get a different sort of answer than you would for other values of n .

¹The quantities a and b might be constants, or they might depend on other variables like r . But we're integrating over θ , so anything that does not depend on θ is treated like a constant.