A Few Useful Integrals

Here is an integral that we’ll see in class and on the homework:

$$\int \frac{dx}{(x^2 + \beta^2)\sqrt{x^2 + \alpha^2}} = \frac{1}{\beta \sqrt{\alpha^2 - \beta^2}} \tan^{-1} \left( \frac{x}{\beta \sqrt{\alpha^2 + x^2}} \right) \tag{1}$$

How do we perform this integral? One way is to use a trig substitution, though it is a little hard to see where it comes from. Consider a right triangle where the side opposite an angle \( \theta \) is \( x\sqrt{\alpha^2 - \beta^2} \), and the side adjacent to the angle is \( \beta\sqrt{x^2 + \alpha^2} \). (Notice that we definitely need \( \alpha^2 > \beta^2 \) so that the length of the opposite side is a \( \mathbb{R} \) number!) Then the hypotenuse has length \( \alpha \sqrt{x^2 + \beta^2} \). Now look at the standard trig functions: \( \sin \theta \), \( \cos \theta \), and \( \tan \theta \). For our right triangle we have

$$\sin \theta = \frac{x\sqrt{\alpha^2 - \beta^2}}{\alpha \sqrt{x^2 + \beta^2}} \tag{2}$$

$$\cos \theta = \frac{\beta \sqrt{x^2 + \alpha^2}}{\alpha \sqrt{x^2 + \beta^2}} \tag{3}$$

$$\tan \theta = \frac{x\sqrt{\alpha^2 - \beta^2}}{\beta \sqrt{x^2 + \alpha^2}} \tag{4}$$

What does this do for us? Well, let’s see what happens when we try to write our integral over \( x \) as an integral over \( \theta \). To convert the integral we need to know how \( dx \) and \( d\theta \) are related. This isn’t too hard – we can use the formula for \( \sin \theta \) or \( \cos \theta \) to figure it out. Let’s use \( \sin \theta \). Taking the derivative of both sides of (2) with respect to \( x \), we wind up with (you should fill in the details here)

$$d\theta = \frac{\beta \sqrt{\alpha^2 - \beta^2}}{(x^2 + \beta^2)\sqrt{x^2 + \alpha^2}} dx \tag{5}$$

$$\Rightarrow dx = \frac{1}{(x^2 + \beta^2)\sqrt{x^2 + \alpha^2}} \cdot d\theta \cdot \frac{1}{\beta \sqrt{\alpha^2 - \beta^2}} \tag{6}$$

That’s great! It leaves us with a very simple integral

$$\int dx \frac{1}{(x^2 + \beta^2)\sqrt{x^2 + \alpha^2}} = \frac{1}{\beta \sqrt{\alpha^2 - \beta^2}} \int d\theta \tag{7}$$

$$= \frac{1}{\beta \sqrt{\alpha^2 - \beta^2}} \theta \tag{8}$$

Now we just express \( \theta \) in terms of our original variable \( x \) using any one of (2), (3), or (4). For instance

$$\frac{1}{\beta \sqrt{\alpha^2 - \beta^2}} \theta = \frac{1}{\beta \sqrt{\alpha^2 - \beta^2}} \tan^{-1} \left( \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \frac{x}{\sqrt{x^2 + \alpha^2}} \right), \tag{9}$$

which is the result (1) we were looking for. When you encounter this integral on the homework you should have \( \alpha^2 > \beta^2 \), so that you get a \( \mathbb{R} \) number.

Another integral you might encounter on homeworks is

$$\int d\theta \frac{\sin \theta}{(a + b \cos \theta)^n} \tag{10}$$
where \( n \) is a number, and \( a \) and \( b \) don’t depend on \( \theta \). This one is much easier to evaluate than the first integral; we just need a simple substitution to turn it into an elementary integral. First let’s make a change of variables to simplify our denominator a bit.

\[
u = a + b \cos \theta \quad du = -b \sin \theta \, d\theta .
\]

This works because the integrand of (10) has the factor of \( \sin \theta \) in its numerator. In the new variable the integral becomes

\[
\int \frac{d\theta}{(a + b \cos \theta)^n} = -\frac{1}{b} \int \frac{du}{u^n} .
\]

I think you can handle it from here! Just remember that the answer you get when \( n \neq 1 \) is qualitatively different than the answer for \( n = 1 \).

Finally, an integral similar to (10) that shows up quite a bit is

\[
\int d\theta \frac{\sin \theta \cos \theta}{(a + b \cos \theta)^n} .
\]

The trick we used for the last integral works here, too. With the change of variables (11) we get

\[
\int d\theta \frac{\sin \theta \cos \theta}{(a + b \cos \theta)^n} = -\frac{1}{b^2} \int du \frac{1}{u^n} (u - a)
\]

where we also used (11) to rewrite the extra factor of \( \cos \theta \) as \((u - a)/b\). We arrive at

\[
\int d\theta \frac{\sin \theta \cos \theta}{(a + b \cos \theta)^n} = \frac{a}{b^2} \int du \frac{1}{u^n} - \frac{1}{b^2} \int du \frac{1}{u^{n-1}} .
\]

This time we need to do two integrals, but they are both elementary. Keep in mind, though, that if \( n = 1 \) or \( n = 2 \) you will get a different sort of answer than you would for other values of \( n \).