A Few Useful Integrals

Here is an integral that we'll see in class and on the homework:

$$\int dx\, \frac{1}{(x^2+\beta^2)\sqrt{x^2+\alpha^2}} = \frac{1}{\beta\sqrt{\alpha^2-\beta^2}}\, \tan^{-1}\left(\frac{\sqrt{\alpha^2-\beta^2}}{\beta}\frac{x}{\sqrt{\alpha^2+x^2}}\right) \tag{1}$$

How do we perform this integral? One way is to use a trig substitution, though it is a little hard to see where it comes from. Consider a right triangle where the side opposite an angle θ is $x\sqrt{\alpha^2-\beta^2}$, and the side adjacent to the angle is $\beta\sqrt{x^2+\alpha^2}$. (Notice that we definitely need $\alpha^2 > \beta^2$ so that the length of the opposite side is a $\mathbb{R}$ number!) Then the hypotenuse has length $\alpha\sqrt{x^2+\beta^2}$. Now look at the standard trig functions: $\sin\theta$, $\cos\theta$, and $\tan\theta$. For our right triangle we have

$$\sin\theta = \frac{x\sqrt{\alpha^2-\beta^2}}{\alpha\sqrt{x^2+\beta^2}} \tag{2}$$

$$\cos\theta = \frac{\beta\sqrt{x^2+\alpha^2}}{\alpha\sqrt{x^2+\beta^2}} \tag{3}$$

$$\tan\theta = \frac{x\sqrt{\alpha^2-\beta^2}}{\beta\sqrt{x^2+\alpha^2}}\,. \tag{4}$$

What does this do for us? Well, let's see what happens when we try to write our integral over x as an integral over θ. To convert the integral we need to know how dx and $d\theta$ are related. This isn't too hard – we can use the formula for $\sin\theta$ or $\cos\theta$ to figure it out. Let's use $\sin\theta$. Taking the derivative of both sides of (2) with respect to x, we wind up with (you should fill in the details here)

$$d\theta = \frac{\beta\sqrt{\alpha^2-\beta^2}}{(x^2+\beta^2)\sqrt{x^2+\alpha^2}}\, dx \tag{5}$$

$$\Rightarrow dx\, \frac{1}{(x^2+\beta^2)\sqrt{x^2+\alpha^2}} = d\theta\, \frac{1}{\beta\sqrt{\alpha^2-\beta^2}} \tag{6}$$

That's great! It leaves us with a very simple integral

$$\int dx\, \frac{1}{(x^2+\beta^2)\sqrt{x^2+\alpha^2}} = \frac{1}{\beta\sqrt{\alpha^2-\beta^2}} \int d\theta \tag{7}$$

$$= \frac{1}{\beta\sqrt{\alpha^2-\beta^2}}\, \theta \tag{8}$$

Now we just express θ in terms of our original variable x using any one of (2), (3), or (4). For instance

$$\frac{1}{\beta\sqrt{\alpha^2-\beta^2}}\, \theta = \frac{1}{\beta\sqrt{\alpha^2-\beta^2}}\, \tan^{-1}\left(\frac{\sqrt{\alpha^2-\beta^2}}{\beta}\frac{x}{\sqrt{\alpha^2+x^2}}\right)\,, \tag{9}$$

which is the result (1) we were looking for. When you encounter this integral on the homework you should have $\alpha^2 > \beta^2$, so that you get a $\mathbb{R}$ number.

Another integral you might encounter on homeworks is

$$\int d\theta\, \frac{\sin\theta}{(a+b\,\cos\theta)^n} \tag{10}$$

where n is a number, and a and b don't depend on θ. This one is much easier to evaluate than the first integral; we just need a simple substitution to turn it into an elementary integral. First let's make a change of variables to simplify our denominator a bit.

$$u = a + b\,\cos\theta \qquad du = -b\,\sin\theta\,d\theta\ . \tag{11}$$

This works because the integrand of (10) has the factor of $\sin\theta$ in its numerator. In the new variable the integral becomes

$$\int d\theta\,\frac{\sin\theta}{(a + b\,\cos\theta)^n} = -\frac{1}{b}\int du\,\frac{1}{u^n}\ . \tag{12}$$

I think you can handle it from here! Just remember that the answer you get when $n \neq 1$ is qualitatively different than the answer for $n = 1$.

Finally, an integral similar to (10) that shows up quite a bit is

$$\int d\theta\,\frac{\sin\theta\,\cos\theta}{(a + b\,\cos\theta)^n} \tag{13}$$

The trick we used for the last integral works here, too. With the change of variables (11) we get

$$\int d\theta\,\frac{\sin\theta\,\cos\theta}{(a + b\,\cos\theta)^n} = -\,\frac{1}{b^2}\int du\,\frac{1}{u^n}\,(u - a) \tag{14}$$

where we also used (11) to rewrite the extra factor of $\cos\theta$ as $(u - a)/b$. We arrive at

$$\int d\theta\,\frac{\sin\theta\,\cos\theta}{(a + b\,\cos\theta)^n} = \frac{a}{b^2}\int du\,\frac{1}{u^n} - \frac{1}{b^2}\int du\,\frac{1}{u^{n-1}}\ . \tag{15}$$

This time we need to do two integrals, but they are both elementary. Keep in mind, though, that if $n = 1$ *or* $n = 2$ you will get a different sort of answer than you would for other values of n.