Problem 1: Interaction energy of two dipoles

Let's figure out the potential energy for a pair of ideal dipoles.

(a) First, show that the energy of an ideal dipole in an electric field is given by

$$U = -\vec{p} \cdot \vec{E}\,.$$

HINT: Start with a physical dipole, with charges q and $-q$ separated by a displacement $\vec{d}$. The potential energy of the charges is $q\,V(\vec{r}+\vec{d}) - q\,V(\vec{r})$. Now write the difference between the potential at $\vec{r}$ and $\vec{r}+\vec{d}$ as an integral of $\vec{E}\cdot d\vec{\ell}$. Next, imagine that $\vec{d}$ is very small, so the integral simplifies, and replace $q\,\vec{d}$ with $\vec{p}$.

(b) In class we worked out the electric field produced by a pure dipole $\vec{p}$ located at the origin. Show that the result we found can be written in the form

$$\vec{E}_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[3\,\hat{r}\,\big(\vec{p}\cdot\hat{r}\big) - \vec{p}\right],$$

We say that an expression like this is "coordinate independent", since it does not refer to a specific choice of coordinates or orientation of the axes.

Combining the two results you just derived gives us the interaction energy of two dipoles $\vec{p}_1$ and $\vec{p}_2$ separated by a displacement $\vec{r}$

$$U = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[\vec{p}_1\cdot\vec{p}_2 - 3\,\big(\vec{p}_1\cdot\hat{r}\big)\,\big(\vec{p}_2\cdot\hat{r}\big)\right].$$

We will use this expression as the starting point for a discussion of dipole-dipole forces.

Problem 2: Potential of a uniformly polarized sphere

Calculate the potential of a uniformly polarized sphere of radius R directly from the formula

$$V(\vec{r}\,) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} d\tau'\,\frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\vec{P}(\vec{r}\,')}{\mathscr{r}^2}\,.$$

Set up your coordinates so that $\vec{P}$ points in the z direction, and express your answer in spherical polar coordinates.

HINT: Since $\vec{P}$ is constant vector, it stays the same as $\vec{r}\,'$ changes. So you can pull it outside of the integral

$$V(\vec{r}\,) = \frac{1}{4\pi\epsilon_0}\,\vec{P}\cdot\int_{\mathcal{V}} d\tau'\,\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,.$$

The integral in this problem should look familiar: it is the same one you would evaluate to get the electric field due to a sphere with constant volume charge density. We never evaluated that integral directly (except for points on the z-axis), but we *did* determine $\vec{E}$ for a uniformly charged sphere using

Gauss's law. So consider our results for $\vec{E}$ both inside and outside a uniformly charged sphere. That's what you should get by evaluating the following expression

$$\frac{\rho}{4\pi\epsilon_0}\int_{\mathcal{V}} d\tau' \frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2} \, .$$

Comparing them will give you the result for the integral inside and outside the sphere.

Problem 3: Spherical shell with fixed polarization

A thick spherical shell with inner radius R_i and outer radius R_o is made of dielectric material with a fixed (or "frozen-in") polarization

$$\vec{P}(\vec{r}\,) = k\, r\, \hat{r}$$

where k is a constant and r is the distance from the center. The shell is electrically neutral – no free charge has been placed on or in the material. Find the electric field in all three regions ($r < R_i$, $R_i < r < R_o$, and $R_o < r$) using the following methods:

(a) Determine all the bound charge and find the field it produces using

$$\oint_S d\vec{a}\cdot\vec{E} = \frac{Q_{\rm enc}}{\epsilon_0} \, .$$

(b) Keeping in mind that there is no *free* charge in this problem, use

$$\oint_S d\vec{a}\cdot\vec{D} = Q_{f,\rm enc}$$

to determine $\vec{D}$, then use $\vec{D} = \epsilon_0\,\vec{E} + \vec{P}$ to find $\vec{E}$.

Problem 4: Field inside a long, cylindrical dielectric

A very long cylinder of linear dielectric material is placed in an otherwise uniform electric field $\vec{E}_0$. The radius of the cylinder is R, its electric susceptibility is χ_e, and its axis is *perpindicular* to $\vec{E}_0$. Find the resulting electric field inside the cylinder.

HINT: This is similar to the dielectric sphere example that we discussed in class. You will need the separation of variables solution to Laplace's equation in cylindrical polar coordinates. Let's say that the axis of the cylinder is along the z-direction. Since the cylinder is very long the potential won't depend on z, and the sum of separable solutions is [1]

$$V(s,\phi) = \sum_{k=1}^{\infty}\left[\left(a_k\, s^k + b_k\, s^{-k}\right)\cos k\phi + \left(c_k\, s^k + d_k\, s^{-k}\right)\sin k\phi\right] ,$$

where k is an integer. [2] Notice that the problem only says that the axis of the cylinder is perpendicular to $\vec{E}_0$. That means that $\vec{E}_0$ could point any direction in what we're calling the x-y plane. You'll have

[1] I encourage you to derive this for yourself – it's not hard. There can also be a constant term, a_0, or a log term, $b_0 \log s$, in the potential. Neither term is needed in this problem, so I have not included them in the expression for $V(s,\phi)$. But in other problems they could be there.

[2] Why must k be an integer? We usually arrive at a condition like that by imposing boundary conditions, but that's not the case here.

to decide how it's oriented when you set up the problem; it's fine if you just assume that $\vec{E}_0$ points in the x direction.

Problem 5: Bound charge on a dielectric cube

A dielectric cube of side L, centered at the origin, carries a frozen-in polarization $\vec{P} = \alpha\, r\, \hat{r}$, where α is a constant. Calculate the total bound charge inside the cube and on its surface, and show that they add up to zero. (You may assume that the bound charge is the same on each of the six sides of the cube, though you should think about why this is. Rather than working in spherical coordinates, try expressing $r\,\hat{r}$ in Cartesian coordinates first!)