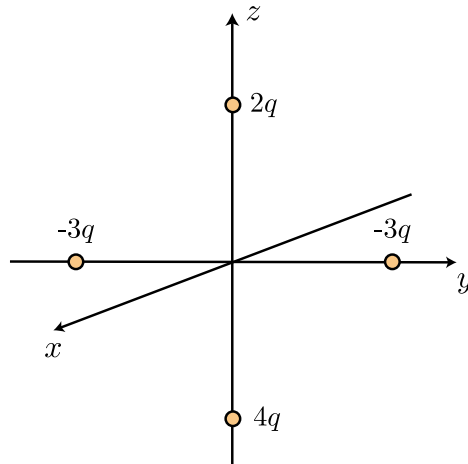


Homework 8: The Multipole Expansion

Due Friday, October 28

Problem 1: Approximate potential

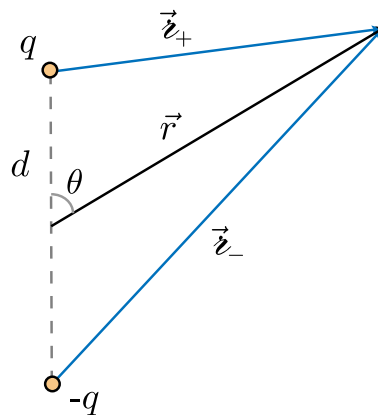
Four particles (one of charge $2q$, one of charge $4q$, and two of charge $-3q$) are placed as shown in the figure below:



Each charge is the same distance a from the origin. Find a simple approximate formula for the potential that is valid at points far from the charges ($r \gg a$). Express your answer in spherical coordinates. (By 'simple approximate formula', I mean the first term in the multipole expansion of the potential that is not zero.)

Problem 2: Quadrupole and Octopole terms for a physical dipole

Two charges, q and $-q$, are separated by a distance d as in the figure below:



The potential at a point \vec{r} is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{z_+} - \frac{q}{z_-} \right). \quad (1)$$

Work out the first four terms in the Taylor expansion of the potential for $r \gg d$. Identify the monopole, dipole, quadrupole, and octopole terms in the expansion. Which ones are zero?

Problem 3: Force on a point charge due to a pure dipole

A “pure” dipole p is situated at the origin, pointing in the z direction.

- (a) What is the force on a point charge q located at $(0, 0, a)$ in Cartesian coordinates?
- (b) What is the force on q if it is located at $(0, a, 0)$?
- (c) How much work does it take to move the charge q from $(0, 0, a)$ to $(0, a, 0)$?

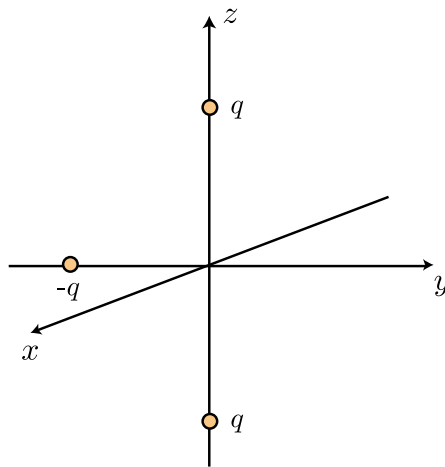
Problem 4: Multipole expansion for a line charge

A thin insulating rod, running from $z = -a$ to $z = +a$, carries a line charge $\lambda(z)$. In the following cases, find the leading term in the multipole expansion of the potential.

- (a) $\lambda(z) = \lambda_0 \cos(\pi z/2a)$
- (b) $\lambda(z) = \lambda_0 \sin(\pi z/a)$
- (c) $\lambda(z) = \lambda_0 \cos(\pi z/a)$

Problem 5: Multipole expansion for three point charges

The figure below shows three point charges, each a distance a from the origin:



Find the approximate *electric field* at points far from the charges. Express your answer in spherical coordinates, and include the contributions from the first two non-zero terms in the multipole expansion of the potential.

Problem 6: The quadrupole moment

In class we worked out the multipole expansion of the potential for an arbitrary distribution of charge. The first two terms in the expansion are given by

$$V_{\text{mon}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int d\tau' \rho(\vec{r}') \quad (2)$$

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int d\tau' \rho(\vec{r}') r' \cos \psi' . \quad (3)$$

As we saw in class they can also be written as

$$V_{\text{mon}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad (4)$$

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \cdot \vec{p}, \quad (5)$$

where the total charge q is the *monopole moment* of the charge distribution, and the vector \vec{p} is the *dipole moment*

$$\vec{p} = \int d\tau' \rho(\vec{r}') \vec{r}' .$$

The third term in the multipole expansion of the potential is the “quadrupole term”. It is given by

$$V_{\text{quad}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int d\tau' \rho(\vec{r}') (r')^2 \left(\frac{3}{2} \cos^2 \psi' - \frac{1}{2} \right) . \quad (6)$$

In this problem we are going to derive a formula similar to (4) and (5) for the quadrupole term.

(a) Show that the quadrupole term in the multipole expansion can be written as

$$V_{\text{quad}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{2r^3} \sum_{i,j=1}^3 \hat{r}_i \hat{r}_j Q_{ij} ,$$

where

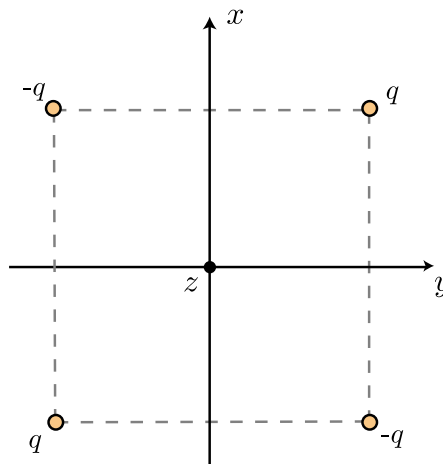
$$Q_{ij} = \int d\tau' \rho(\vec{r}') [3 r'_i r'_j - (r')^2 \delta_{ij}] . \quad (7)$$

In these expressions a subscript i or j refers to components of vectors and matrices, with the values 1, 2, and 3 indicating the x , y , and z directions, respectively. The quantity δ_{ij} is known as the *Kronecker delta*. You may think of it as the 3×3 identity matrix, so

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} .$$

The 3×3 matrix Q_{ij} is called the *quadrupole moment* of the charge distribution¹.

(b) Find all nine components of Q_{ij} for the configuration of charges shown below.



¹Actually, δ_{ij} and Q_{ij} aren't matrices, though it seems natural to think of them that way. They are examples of a mathematical object called a *tensor*. A matrix is a particular kind of tensor.

The square has sides of length a (not $2a$!) and lies in the xy plane, with its center at the origin.

- (c) Show that the quadrupole moment is independent of the placement of the origin if the monopole and dipole moments both vanish.

Hint: Suppose you move the origin, so that the separation vector between the old origin and the new origin is \vec{b} . Then the position vectors in (7) change from \vec{r}' to $\vec{r}' - \vec{b}$. Now what happens if you calculate Q_{ij} ?