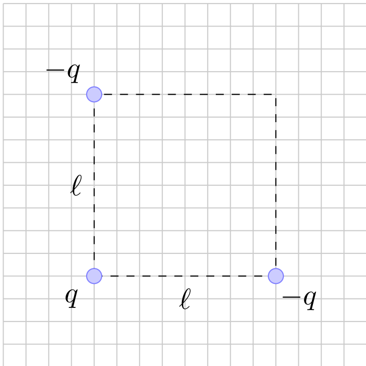
Homework 5: Electrostatic Potential Energy

Due: Friday, September 30

Problem 1: Point charges at the corners of a square

Three charges are situated at the corners of a square with sides of length ℓ, as shown in the figure below. How much work must one do to bring a fourth charge $+q$ in from infinitely far away and place it on the unoccupied corner? How much work is required to bring all four charges together?



Problem 2: A uniform solid sphere

Find the electrostatic potential energy stored in a solid sphere of radius R with a total charge q distributed uniformly throughout its volume. Do this in the following three ways:

(a) First, use what you know about the potential inside a uniformly charged sphere to evaluate

$$W = \frac{1}{2}\int_{\mathcal{V}} \mathrm{d}\tau\, \rho\, V \ , \tag{1}$$

where the integration volume $\mathcal{V}$ is just the sphere itself.

(b) Second, repeat our calculation from class where we integrated $|\vec{E}|$ over all space

$$W = \frac{\varepsilon_0}{2}\int_{\text{A.S.}} \mathrm{d}\tau\, |\vec{E}|^2 \ . \tag{2}$$

I understand that you've already seen this worked out, but it's important to go through the details yourself.

(c) Finally, evaluate the "intermediate" expression

$$W = \frac{\varepsilon_0}{2}\int_{\mathcal{V}} \mathrm{d}\tau\, |\vec{E}|^2 + \frac{\varepsilon_0}{2}\oint_{\mathcal{S}} \mathrm{d}\vec{a}\cdot\vec{E}\, V \ , \tag{3}$$

but use the region inside a sphere of radius $a > R$ as the volume $\mathcal{V}$ (so $\mathcal{S}$ is the sphere $r = a$).

You should of course get the same result in all three cases. In the last case, what happens to the two contributions (the surface integral and volume integral) as $a \to \infty$?

Problem 3: A solid sphere with quadratic charge density

A solid sphere of radius R carries a charge density $\rho(r) = \alpha\, r^2$ (where α is a constant with appropriate units). Find the energy of the configuration. Check your answer by calculating it two different ways.

Problem 4: Concentric spherical shells

Consider two uniformly charged, concentric spherical shells. The inner shell, with radius R_1, carries a net charge $-q$. The outer shell, with radius R_2, carries a net charge q. Calculate the energy of this configuration in the following ways:

(a) Use formula (2) from the previous problem.

(b) Evaluate the total work needed to assemble this configuration

$$W_{\text{tot}} = W_1 + W_2 + \varepsilon_0 \int_{\text{A.S.}} \mathrm{d}\tau\, \vec{E}_1 \cdot \vec{E}_2 \ . \tag{4}$$

where $\vec{E}_1$ and $\vec{E}_2$ are the electric fields produced by the inner and outer spheres, respectively. The last term in (4) is the 'interaction energy'. How would you interpret this quantity?

Problem 5: $1/r^2$ Forces

This problem is a little outside the scope of the class, but it's an interesting exercise.

Coulomb's Law and Newton's Universal Law of Gravitation both describe forces that vary with distance as $1/r^2$. As a result, they share many common properties.

(a) Show that the gravitational field for a hollow, uniform shell with mass M and radius R is the same as the field for a point mass if $r > R$, but zero if $r < R$. That is, evaluate the right-hand-side of

$$\vec{g} = -G \frac{M}{4\pi R^2} \oint_{\mathcal{S}} da' \, \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \ , \tag{5}$$

where the integral is over the surface of a sphere with radius R, and the factor of $M/4\pi R^2$ is the (surface) mass density of the shell. Since Coulomb's Law and Newton's Law have the same form, it isn't surprising that we get the same result for the electric field of a hollow shell with uniform charge density. This isn't a special property of electrostatics or gravity – it's a property of any $1/r^2$ force.

(b) Of course, if you were calculating the electric field for something as symmetric as a uniform charged shell you wouldn't bother evaluating the integral – you'd just use Gauss's Law! It must be that there is a Gauss's Law for gravity, too. Consider the integral formulas for the gravitational and electric fields

$$\vec{g} = -G \int_{\mathcal{V}} d\tau' \, \rho_m(\vec{r}\,') \, \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad\qquad \vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} d\tau' \, \rho(\vec{r}\,') \, \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \ , \tag{6}$$

where G is Newton's Constant, ρ_m is mass density, and (as usual) ρ is charge density. How would you write the integral and differential forms of Gauss's Law for gravity?

(c) Would we get the same result in part (a) if Coulomb's Law (or Newtons's Law) involved a $1/r$ force instead of a $1/r^2$ force? What do you obtain for $\vec{F}$ in that case?