Homework 3: Orthogonal Coordinate Systems, Velocity and Acceleration
Due Monday, February 5

Problem 1: Velocity and acceleration in SPC

Using your results from the previous homework, derive expressions for the velocity ($\dot{\vec{r}}$) and acceleration ($\ddot{\vec{r}}$) vectors in spherical polar coordinates.

Problem 2: Line integrals in polar coordinates

Express the vector $\vec{A} = x^2 y\,\hat{x} + y^2 x\,\hat{y}$ entirely in polar coordinates (including the unit vectors!), and then evaluate the integral of $\vec{A} \cdot d\vec{\ell}$ along the following paths:

(a) From $\phi = 0$ to $\phi = \pi/2$ along the path $\rho = 1$.

(b) From $\rho = 1$ to $\rho = 3$ along the path $\phi = \pi/4$.

Problem 3: Elliptical cylindrical coordinates

Elliptical cylindrical coordinates describe points using a grid composed of ellipses and hyperbolae in the x-y plane, along with the usual z coordinate. They are related to Cartesian coordinates by

$$x = a\,\cosh\mu\,\cos\nu \tag{1}$$

$$y = a\,\sinh\mu\,\sin\nu \tag{2}$$

$$z = z\ , \tag{3}$$

where a is a positive number, $0 \le \mu < \infty$, and $0 \le \nu < 2\pi$. Derive the scale factors and unit vectors for these coordinates.

HINT: If you haven't seen cosh and sinh in a while, they are defined as

$$\cosh\mu = \frac{1}{2}\left(e^{\mu} + e^{-\mu}\right) \qquad \sinh\mu = \frac{1}{2}\left(e^{\mu} - e^{-\mu}\right)\ . \tag{4}$$

Since $d(e^{\pm\mu})/d\mu = \pm e^{\pm\mu}$, their derivatives are

$$\frac{d\cosh\mu}{d\mu} = \sinh\mu \qquad \frac{d\sinh\mu}{d\mu} = \cosh\mu\ . \tag{5}$$

You can also check that $\cosh^2\mu - \sinh^2\mu = 1$ by plugging in the expressions above and expanding. So cosh and sinh are like cos and sin, with plusses where you'd expect minuses and vice-versa.

Problem 4: Celestial mechanics

In class we looked at Newton's laws applied to the problem of a planet orbiting a star. To a pretty good approximation, the planet's distance $r(t)$ from the star satisfies the equation

$$\ddot{r}(t) - \frac{j^2}{r(t)^3} = -\frac{M\,G}{r(t)^2}\ , \tag{6}$$

where j is the planet's angular momentum divided by its mass, M is the mass of the star, and G is Newton's constant. The planet's angular momentum $J = m\,r(t)^2\,\dot{\phi}(t)$ stays the same throughout the orbit, so $j = J/m = r(t)^2\,\dot{\phi}(t)$ is constant. We saw in class that we could rewrite this as an equation for $r(\phi)$ instead of $r(t)$. Using the chain rule and the relationship between $\dot{\phi}$, r, and j, the time derivative of r can be written as

$$\dot{r}(t) = \frac{dr}{dt} = \frac{d\phi}{dt}\,\frac{dr}{d\phi} = \frac{j}{r^2}\,\frac{dr}{d\phi}\ . \tag{7}$$

Show that once we rewrite equation (6) this way [1] and apply the substitution $r(\phi) = 1/u(\phi)$, we wind up with the much simpler equation

$$u''(\phi) + u(\phi) = \frac{M\,G}{j^2}\ , \tag{8}$$

where $'$ indicates a derivative with respect to ϕ.

[1] You'll have to apply the chain rule twice for the $\ddot{r}(t)$ term!