Homework 2: Spherical Polar Coordinates

Due Monday, January 29

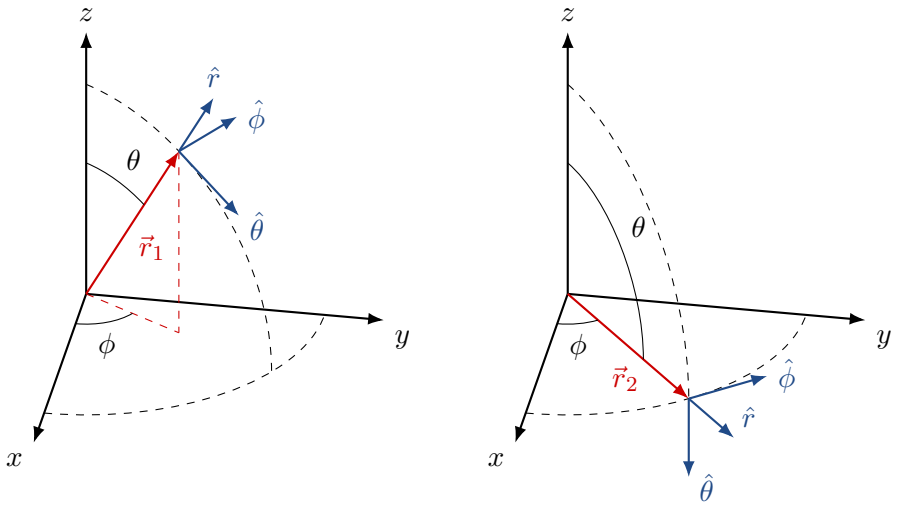
Problem 1: Spherical Polar Coordinates

Cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ) are related by

$$x = r\,\sin\theta\,\cos\phi \qquad y = r\,\sin\theta\,\sin\phi \qquad z = r\,\cos\theta\,. \tag{1}$$

Here $r \geq 0$ is the distance from the origin, $0 \leq \theta \leq \pi$ is the angle down from the z-axis, and $0 \leq \phi < 2\pi$ is the counterclockwise angle from the x-axis in the x-y plane. Notice that $\phi = 2\pi$ means the same thing as $\phi = 0$.

In Cartesian coordinates, the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ point in the directions of increasing x, y, and z, respectively. The same is true for the unit vectors in spherical polar coordinates: $\hat{r}$ points in the direction of increasing r, $\hat{\theta}$ in the direction of increasing θ, etc. In the figures below, you can see that at different points the directions associated with $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ change relative to $\hat{x}$, $\hat{y}$, and $\hat{z}$.



Notice, however, that these unit vectors remain perpendicular to each other, and their relative orientation does not changes.

Let's work out the basic properties of spherical polar coordinates and their unit vectors.

(a) Determine the scale factors h_r, h_θ, and h_ϕ by expressing ds^2 in spherical polar coordinates.

(b) Derive expressions for the unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ as combinations of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. It's okay to write the coefficients of the Cartesian vectors in terms of r, θ and ϕ; you don't have to express them as functions of x, y, and z.

(c) Invert the results you got in the last part, expressing $\hat{x}$, $\hat{y}$, and $\hat{z}$ as combinations of the vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$.

(d) Express the position vector $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$ completely in spherical polar coordinates. That is, a vector that uses the spherical polar coordinate basis vectors, with any coefficients written as functions of r, θ, and/or ϕ.

Be sure to show all your work in each part. Writing down the correct answers is not sufficient – you must show how you obtained them.

Problem 2: Vectors in Spherical Polar Coordinates

Suppose I tell you about a vector by describing its components in Cartesian coordinates:

$$\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}\ . \tag{2}$$

In spherical polar coordinates we would describe that same vector by specifying its components in the r, θ, and ϕ directions:

$$\vec{A} = A_r\,\hat{r} + A_\theta\,\hat{\theta} + A_\phi\,\hat{\phi}\ . \tag{3}$$

Work out expressions for the components A_r, A_θ, and A_ϕ in terms of the Cartesian components A_x, A_y, A_z and the spherical polar coordinate scale factors.