Homework 2: More Vector Analysis

Due: Thursday, September 8^{th}

Important: You may not use *Mathematica* or similar software on this homework. In fact, on most homework assignments you should assume that *Mathematica* is not allowed. It is really important that you learn how to handle the math on these assignments without resorting to computers. There is a lot of understanding to be uncovered in working out the details for yourself.

Problem 1: Spherical Polar Coordinates

Cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ) are related by

$$x = r\,\sin\theta\,\cos\phi \qquad y = r\,\sin\theta\,\sin\phi \qquad z = r\,\cos\theta\ . \tag{1}$$

(a) Derive expressions for the unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ in terms of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$.

(b) Now invert your results from part (a) and derive expressions for $\hat{x}$, $\hat{y}$, and $\hat{z}$ as appropriate combinations of $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$.

(c) Use your results from part (b) to derive expressions for the components of a vector in spherical polar coordinates. That is, give the components of $\vec{V} = V_r\,\hat{r} + V_\theta\,\hat{\theta} + V_\phi\,\hat{\phi}$ in terms of V_x, V_y, and V_z.

(d) Derive Griffiths' equation (1.70) for the gradient of a function in spherical polar coordinates.

(e) Derive Griffiths' equation (1.71) for the divergence of a vector in spherical polar coordinates.

(f) Combine your results from (d) and (e) to derive Griffiths' expression (1.73) for the Laplacian of a function in spherical polar coordinates.

Problem 2: Why We Love Conservative Forces

An object is moving in the x-y plane along the path $y = x^2 + x - 2$. It experiences a force

$$\vec{F}(x, y) = \frac{2\,\alpha\,x}{(x^2+y^2)^2}\,\hat{x} + \frac{2\,\alpha\,y}{(x^2+y^2)^2}\,\hat{y}\ , \tag{2}$$

where α is a constant. What is the work performed by this force as the object moves from the point $(2, 4)$ to the point $(4, 18)$?

(*Hint:* This problem is messy if you try to integrate $\vec{F}\cdot d\vec{\ell}$ directly. It can be done, but there is a much easier way – think about the title of the problem.)

Problem 3: Divergence Theorem

The components of a vector function $\vec{A}$ are given in cylindrical polar coordinates $\{s, \phi, z\}$ as

$$\vec{A} = \frac{s^2}{L}\,\sin\left(\frac{z}{L}\right)\,\hat{s} + s\,\cos\left(\frac{z}{L}\right)\,\hat{z}\ , \tag{3}$$

where L is a constant with units of length.

(a) Use a surface integral to calculate the flux of $\vec{A}$ through the closed surface bounding the region $2\,L \le s \le 4\,L$, $0 \le z \le \pi\,L$, $0 \le \phi < 2\pi$.

(b) Evaluate the same integral using the divergence theorem.

Problem 4: Stokes' Theorem

The components of a vector function $\vec{V}$ are given in Cartesian coordinates as

$$\vec{V} = -\frac{y^2}{2\,a^2}\,\hat{x} + \left(\frac{y\,z^2}{2\,a^3} - \frac{x\,y}{a^2}\right)\hat{y} + \left(\frac{y^2 z}{2\,a^3} + 1\right)\hat{z}\ , \tag{4}$$

where a is a constant. Use Stokes' theorem to calculate the integral

$$\oint_P \vec{V} \cdot \mathrm{d}\vec{\ell} \tag{5}$$

where P is the rectangle $ABCD$ with corners $A = (-1, 2, 0)$, $B = (3, 2, 0)$, $C = (3, 5, 0)$, and $D = (-1, 5, 0)$.

Problem 5: The Three-Dimensional Delta Function

Perform the following integrals involving the three-dimensional delta function:

(a)

$$\int_{\substack{\text{all}\\ \text{space}}} (r^2 - \vec{r} \cdot \vec{a} + a^2)\ \delta^3(\vec{r} - \vec{a}\,)\ d\tau\ ,$$

where $\vec{a}$ is a constant vector with magnitude a.

(b)

$$\int_{\mathcal{V}} \left|\vec{b} - \vec{r}\,\right|^2\ \delta^3(4\,\vec{r}\,)\ d\tau\ ,$$

where $\mathcal{V}$ is a cube of side 3, centered on the origin, and $\vec{b} = 3\,\hat{x} + 5\,\hat{z}$.

Problem 6: The Helmholtz Theorem

Consider the following vector functions:

$$\vec{F}_1 = y\,\hat{x} + z\,\hat{y} + 0\,\hat{z} \tag{6}$$

$$\vec{F}_2 = x^2\,\hat{x} - y^2\,\hat{y} + z^2\,\hat{z} \tag{7}$$

$$\vec{F}_3 = -y\,z\,\hat{x} - z\,x\,\hat{y} - x\,y\,\hat{z} \tag{8}$$

(a) Calculate the divergence and curl of $\vec{F}_1$ and $\vec{F}_2$.

(b) Which one of $\vec{F}_1$ and $\vec{F}_2$ can be written as the gradient of a scalar function? Find a scalar potential that does the job.

(c) Which one of $\vec{F}_1$ and $\vec{F}_2$ can be written as the curl of a vector function? Find a suitable vector potential.

(d) Show that $\vec{F}_3$ can be written both as the gradient of a scalar function and as the curl of a vector function.

(e) Find scalar and vector potentials for $\vec{F}_3$.