Homework 11: Magetostatics and Ampère's Law

Optional for Tuesday, November 21

Problem 1: Cylinder with volume and surface currents

A long, straight cylinder of radius R carries a uniform current density $\vec{J} = J\,\hat{z}$ on its interior, and a surface current $\vec{K} = -K\,\hat{z}$ on its exterior. Use Ampère's law to find the magnetic field both inside and outside the cylinder. You may treat the cylinder as if it were infinitely long.

Problem 2: Magnetic field in a coaxial cable

A very long coaxial cable consists of an inner conductor and an outer conductor that carry current in opposite directions. The inner conductor is a cylinder ($0 \leq s \leq a$) with uniform current density $\vec{J}_i = J\,\hat{z}$, while the outer conductor is a cylindrical shell ($b \leq s \leq c$) with uniform current density $\vec{J}_o = -J\,\hat{z}$. The region between the inner and outer conductors ($a < s < b$) is empty. Use Ampère's law to find the magnetic field in the regions (i) $s < a$, (ii) $a < s < b$, (iii) $b < s < c$, and (iv) $s > c$.

Caution: In the first problem I did not say anything specific about J and K, and in the second problem $\vec{J}_o = -\vec{J}_i$ but the radii a, b, and c are arbitrary. So don't assume that the net current is zero in either problem!

Problem 3: Ampèrian loops and surfaces

The integral form of Ampère's law states that the current I_{enc} passing through an open surface $\mathcal{S}$ is related to the integral of $d\vec{\ell}\cdot\vec{B}$ around the surface's perimeter

$$\oint_{\mathcal{P}} d\vec{\ell}\cdot\vec{B} = \mu_0\, I_{\text{enc}} \qquad \text{with} \qquad I_{\text{enc}} = \int_{\mathcal{S}} d\vec{a}\cdot\vec{J}\,. \tag{1}$$

When applying Ampère's law, we consider what we know about the magnetic field and then look for an "Ampèrian loop" $\mathcal{P}$ that will make the integral on the left-hand side of (1) easy to evaluate. Once we have chosen a loop $\mathcal{P}$, the surface $\mathcal{S}$ that we use to calculate I_{enc} usually seems obvious. For example, if we chose a circle for $\mathcal{P}$ we would probably assume that $\mathcal{S}$ is the flat disk inside the circle. But there are many different surfaces $\mathcal{S}$ that all have the same perimeter $\mathcal{P}$ – how do we know which one to use when determining I_{enc}? The answer is that it does not matter! As long as the current is steady we should get the same I_{enc} using any surface $\mathcal{S}$ whose perimeter is $\mathcal{P}$. Convince yourself that this is at least plausible by calculating the enclosed current for the loop $x^2 + y^2 = R^2$ in the x-y plane and a uniform current density $\vec{J} = J\,\hat{z}$, using the following two surfaces

$$\begin{aligned} &(i) \quad \mathcal{S}_1 \,:\, x^2 + y^2 \leq R^2\,, z = 0 \\ &(ii) \quad \mathcal{S}_2 \,:\, x^2 + y^2 + z^2 = R^2\,, z \geq 0\,. \end{aligned}$$

The first surface ($\mathcal{S}_1$) is a disk, and the second surface ($\mathcal{S}_2$) is a hemisphere.