Problem 1: Dot products and angles

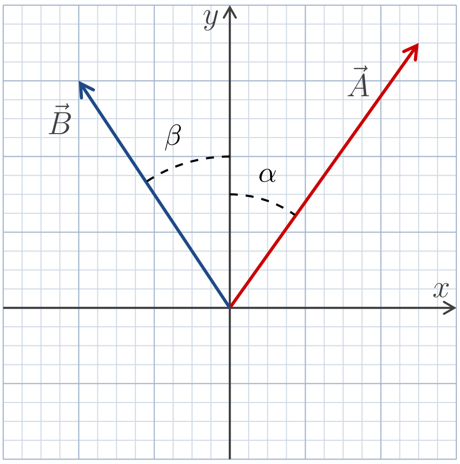
Use what you know about the dot product of two vectors to determine the angle, in radians, between the following pairs of vectors.

(a) $\vec{A}_1 = 4\,\hat{x} + 3\,\hat{y}$ and $\vec{A}_2 = -2\,\hat{x} - 3\,\hat{y}$

(b) $\vec{B}_1 = 17\,\hat{x} + 11\,\hat{y} - 13\,\hat{z}$ and $\vec{B}_2 = 9\,\hat{x} - 20\,\hat{y} + 11\,\hat{z}$

Problem 2: Vector products and trig identities

Consider the vectors $\vec{A}$ and $\vec{B}$ (with magnitudes A and B, respectively) oriented as shown in the diagram below:



(a) Use the dot product $\vec{A}\cdot\vec{B}$ to show that $\cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$.

(b) Use the cross product $\vec{A}\times\vec{B}$ to show that $\sin(\alpha+\beta) = \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta$.

Problem 3: Forces, line integrals, and work

Suppose an object is experiencing a force

$$\vec{F} = 10\,\frac{\mathrm{N}}{\mathrm{m}}\,y\,\hat{x} - 2\,\frac{\mathrm{N}}{\mathrm{m}}\,x\,\hat{y}\;. \tag{1}$$

Find the work performed on the object by this force if it moves from the point $(0,0)$ to the point $(3\,\mathrm{m}, 27\,\mathrm{m})$ along the path

$$y(x) = 3\,x + \frac{2}{\mathrm{m}}\,x^2\;. \tag{2}$$

Remember, the work performed by a force $\vec{F}$ on an object moving from A to B along path P is

$$W = {}_P\!\int_A^B d\vec{\ell}\cdot\vec{F}\;, \tag{3}$$

where $d\vec{\ell} = dx\,\hat{x} + dy\,\hat{y}$ is the infinitesimal displacement from point-to-point along the path. In this problem, you will need to express dy in terms of dx for the path described above.

Problem 4: Surface integrals

Suppose S is a flat, rectangular surface in the y-z plane ($x = 0$) with $-2 \le y \le 2$ and $-3 \le z \le 5$. Evaluate the surface integral

$$\int_S d\vec{A} \cdot \vec{B} \tag{4}$$

for the following vectors:

(a) $\vec{B} = 3\,\hat{x}$

(b) $\vec{B} = -7\,\hat{x}$

(c) $\vec{B} = y^2\,\hat{x} - x\,y\,\hat{y} + x^2\,\hat{z}$

(d) Find a non-zero vector $\vec{B}$ that gives zero for this integral.

You should use $\hat{x}$ as the normal direction in the vector area element $d\vec{A}$. (You could just as well use $-\hat{x}$, but to make things easier on the grader let's all use $\hat{x}$.)

Problem 5: Polar coordinates

In our discussion of Orthogonal Coordinate Systems we will frequently use polar coordinates (ρ, ϕ) as an example. They are related to the Cartesian coordinates (x, y) by

$$x = \rho\,\cos\phi \qquad y = \rho\,\sin\phi\ . \tag{5}$$

Work out expressions for ρ and ϕ as functions of x and y.