Homework 1: Vector Analysis

Due Tuesday, August 30th

This assignment is due at our first class, on August 30^{th}.

This is the first homework assignment for Physics 351. It is a review to make sure you are up to speed on topics covered in Physics 301 (Math Methods). We will spend the first week reviewing some of the more complicated topics, but you should already know the material covered in this assignment when class starts. There are two important rules for this assignment:

- You may not use MATHEMATICA (or Maple, Matlab, Wolfram Alpha, etc) on any of these problems. Everything must be done by hand.
- You should do these problems on your own. Copying a solution found online or in a solutions manual defeats the purpose of the assignment and will result in a grade of zero.

Think *very carefully* about the second point: this is a review of skills that are absolutely essential for this class. You really need to make sure you are comfortable with this stuff before we get started, and the only way to do that is to work these problems out yourself. If you have questions you can email me at rmcnees@luc.edu. You can also stop by my office if you're around over the summer!

Problem 1: Div, Grad, Curl Identities

Let $f(x, y, z)$ and $g(x, y, z)$ be scalar functions, and let $\vec{A}(x, y, z)$ and $\vec{B}(x, y, z)$ be vector functions. Prove the following identities involving divergence, gradient, and curl.

$$\vec{\nabla}(f+g) = \vec{\nabla}f + \vec{\nabla}g \tag{1}$$

$$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B} \tag{2}$$

$$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B} \tag{3}$$

$$\vec{\nabla}(fg) = g\,\vec{\nabla}f + f\,\vec{\nabla}g \tag{4}$$

$$\vec{\nabla}\cdot(f\,\vec{A}) = f\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f \tag{5}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \tag{6}$$

$$\vec{\nabla}\times(f\,\vec{A}) = \vec{\nabla}f\times\vec{A} + f\,\vec{\nabla}\times\vec{A} \tag{7}$$

In this case, "prove" means you should show enough steps so that someone looking over your work can tell that you convinced yourself that each one of these identities is correct.

Problem 2: The Vector Triple Product

The triple product rule for three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ is

$$\vec{A}\times\left(\vec{B}\times\vec{C}\,\right) = \vec{B}\left(\vec{A}\cdot\vec{C}\,\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\,\right)\,. \tag{8}$$

Prove this identity by writing out both sides of the equation in components and showing that they are the same.

Problem 3: Building a Plane out of a Point and two Vectors

Suppose $\vec{A}$ and $\vec{B}$ are two vectors that point in different directions. If a vector $\vec{n}$ is orthogonal to both $\vec{A}$ and $\vec{B}$, then it must be normal to any plane parallel to $\vec{A}$ and $\vec{B}$. There are many such planes, but only one of them will pass through a given point $\mathbf{P}$. Give an equation describing the plane parallel to the vectors $\vec{A} = -1\,\hat{x} - 2\,\hat{y} + 1\,\hat{z}$ and $\vec{B} = -1\,\hat{x} + 3\,\hat{y} - 4\,\hat{z}$ and passing through the point $(1, 1, 2)$. (*Hint:* The separation vector between $\mathbf{P}$ and any other point (x, y, z) in the plane is orthogonal to $\vec{n}$.)

Problem 4: Gradients, Normal Vectors

Consider the scalar function $h(x, y, z)$ given by

$$h(x, y, z) = \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z}{c}\ . \tag{9}$$

All the points (x, y, z) that satisfy the equation $h(x, y, z) = 0$ form a surface known as a *hyperbolic paraboloid.*

(a) Find the gradient of this function.

(b) Use your answer to obtain a unit vector that is normal to the surface.

(c) What is the unit normal vector at the point $(a/\sqrt{3}, 2\,b/\sqrt{3}, c)$?

Problem 5: Divergence and Curl

Compute the divergence and curl of the following vectors:

(a) $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$.

(b) The unit vector $\hat{r} = \dfrac{\vec{r}}{||\,\vec{r}\,||}$

(c) $\vec{V} = z\,\hat{x} + x\,\hat{y} + y\,\hat{z}$

(d) $\vec{w} = -x^2\,y\,\hat{x} + y^2\,x\,\hat{y} + x\,y\,z\,\hat{z}$

Problem 6: Position Vector

Suppose $\vec{k} = k_x\,\hat{x} + k_y\,\hat{y} + k_z\,\hat{z}$ is a constant vector, and $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$ is the position vector of a point with coordinates (x, y, z). Evaluate the following:

(a) $\vec{\nabla} \times (\,\vec{k} \times \vec{r}\,)$

(b) $\vec{\nabla}(\,\vec{k} \cdot \vec{r}\,)$

Problem 7: Line Integrals

Given the vector $\vec{V} = x\,y\,\hat{x} - \frac{3}{2}\,y^2\,\hat{y}$, evaluate the line integral

$$\int_{(0,\,0)}^{(3,\,3)} \vec{V} \cdot \mathrm{d}\vec{\ell} \tag{10}$$

from the point $(0, 0)$ to the point $(3, 3)$, along the following paths:

(a) $y = x^2 - 2\,x$

(b) Along $y = 0$ from $x = 0$ to $x = 3$, then along $x = 3$ from $y = 0$ to $y = 3$.

(c) Along $x = 0$ from $y = 0$ to $y = 3$, then along $y = 3$ from $x = 0$ to $x = 3$.

In this problem we're considering paths in the x-y plane; there is no z direction to worry about. If you don't yet have a copy of the textbook, $\mathrm{d}\vec{\ell}$ is the notation Griffiths uses for the vector $\mathrm{d}x\,\hat{x} + \mathrm{d}y\,\hat{y}$. **Remember**: this is a *line* integral. Take the time to review line, surface, and volume integrals if you aren't clear on what each one is and how they are different!