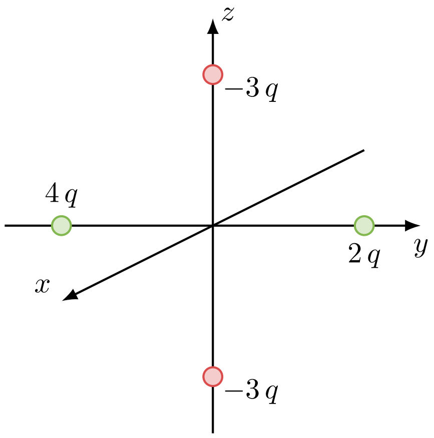
Problem 1: Approximate potential

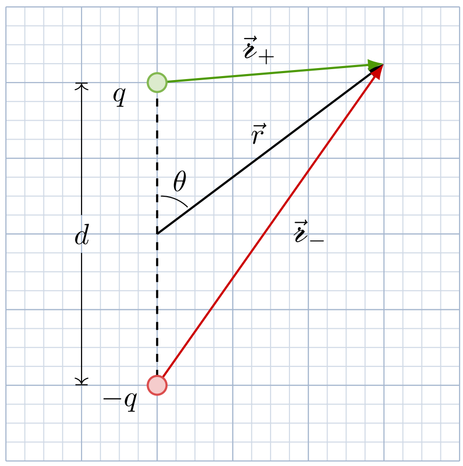
Four particles (one of charge $2q$, one of charge $4q$, and two of charge $-3q$) are placed as shown in the figure below:



Each charge is the same distance L from the origin. Find a simple approximate formula for the potential that is valid at points far from the charges ($r \gg L$). Express your answer in spherical coordinates. (By 'simple approximate formula', I mean the first term in the multipole expansion of the potential that is not zero.)

Problem 2: Quadrupole and Octopole terms for a physical dipole

Two charges, q and $-q$, are separated by a distance d as in the figure below:



The potential at a point $\vec{r}$ is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right) . \tag{1}$$

Work out the first four terms in the Taylor expansion of the potential for $r \gg d$. Identify the monopole, dipole, quadrupole, and octopole terms in the expansion. Which ones are zero?

Problem 3: Force on a point charge due to a pure dipole

A "pure" dipole p is situated at the origin, pointing in the z direction.

(a) What is the force on a point charge q located at $(0, 0, d)$ in Cartesian coordinates?

(b) What is the force on q if it is located at $(d, 0, 0)$?

(c) How much work does it take to move the charge q from $(0, 0, d)$ to $(d, 0, 0)$?

Problem 4: Multipole expansion for a line charge

A thin insulating rod, running from $z = -a$ to $z = +a$, carries a line charge $\lambda(z)$. In the following cases, find the leading term in the multipole expansion of the potential.

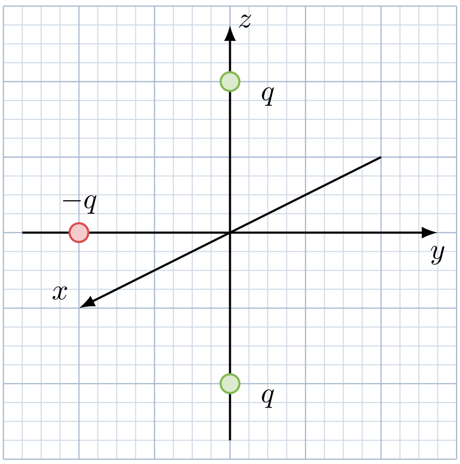
(a) $\lambda(z) = \lambda_0 \cos(\pi z/2a)$

(b) $\lambda(z) = \lambda_0 \sin(\pi z/a)$

(c) $\lambda(z) = \lambda_0 \cos(\pi z/a)$

Problem 5: Multipole expansion for three point charges

The figure below shows three point charges, each a distance L from the origin:



Find the approximate *electric field* at points far from the charges. Express your answer in spherical coordinates, and include the contributions from the first two non-zero terms in the multipole expansion of the potential.

Problem 6: The quadrupole moment

In class we worked out the multipole expansion of the potential for an arbitrary distribution of charge.

The first two terms in the expansion are given by

$$V_{\text{mon}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r}\int d\tau' \rho(\vec{r}\,') \tag{2}$$

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r^2}\int d\tau' \rho(\vec{r}\,')\, r' \cos\psi' \ . \tag{3}$$

As we saw in class they can also be written as

$$V_{\text{mon}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r} \tag{4}$$

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r^2}\,\hat{r}\cdot\vec{p} \ , \tag{5}$$

where the total charge q is the *monopole moment* of the charge distribution, and the vector $\vec{p}$ is the *dipole moment*

$$\vec{p} = \int d\tau' \rho(\vec{r}\,')\,\vec{r}\,' \ . \tag{6}$$

The third term in the multipole expansion of the potential is the "quadrupole term". It is given by

$$V_{\text{quad}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r^3}\int d\tau' \rho(\vec{r}\,')\,(r')^2 \left(\frac{3}{2}\cos^2\psi' - \frac{1}{2}\right) \ . \tag{7}$$

In this problem we are going to derive a formula similar to (4) and (5) for the quadrupole term.

(a) Show that the quadrupole term in the multipole expansion can be written as

$$V_{\text{quad}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{2\,r^3}\sum_{i,j=1}^{3}\hat{r}_i\,\hat{r}_j\,Q_{ij} \ ,$$

where

$$Q_{ij} = \int d\tau'\,\rho(\vec{r}\,')\left[3\,r_i'\,r_j' - \left(r'\right)^2\delta_{ij}\right] \ . \tag{8}$$

In these expressions a subscript i or j refers to components of vectors and matrices, with the values 1, 2, and 3 indicating the x, y, and z directions, respectively. The quantity δ_{ij} is known as the *Kronecker delta.* You may think of it as the 3×3 identity matrix, so

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \ .$$

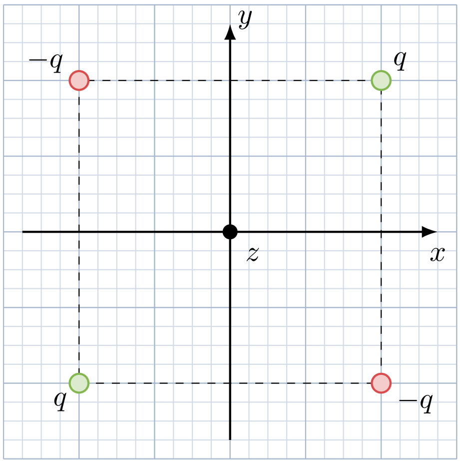
The 3×3 matrix Q_{ij} is called the *quadrupole moment* of the charge distribution [1].

HINT: If the notation here isn't clear, the dipole moment (6) would be written as

$$p_i = \int d\tau' \rho(\vec{r}\,')\, r_i' \ . \tag{9}$$

(b) Find all nine components of Q_{ij} for the configuration of charges shown below.

[1] Actually, δ_{ij} and Q_{ij} aren't matrices, though it seems natural to think of them that way. They are examples of a mathematical object called a *tensor*. A matrix is a particular kind of tensor.



The square has sides of length L (not $2L$!) and lies in the x-y plane, with its center at the origin.

(c) Show that the quadrupole moment is independent of the placement of the origin if the monopole and dipole moments both vanish.

HINT: Suppose you move the origin, so that the separation vector between the old origin and the new origin is $\vec{b}$. Then the position vectors in (8) change from $\vec{r}\,'$ to $\vec{r}\,' - \vec{b}$. Now what happens if you calculate Q_{ij}?