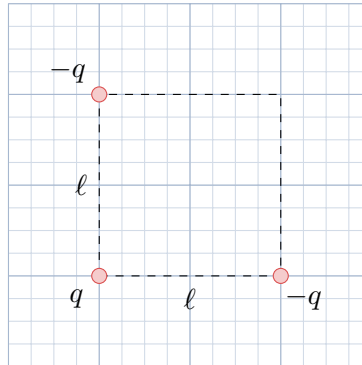


Homework 5: Electrostatic Potential Energy

Due: Monday, October 1

Problem 1: Point charges at the corners of a square

Three charges are situated at the corners of a square with sides of length ℓ , as shown in the figure below. How much work must one do to bring a fourth charge $+q$ in from infinitely far away and place it on the unoccupied corner, in the presence of the other three charges? How much work is required to bring all four charges together from infinitely far away?



Problem 2: A solid sphere with quadratic charge density

A solid sphere of radius R carries a charge density $\rho(r) = \alpha r^2/R^2$, where α is a constant with appropriate units. Find the energy of this distribution of charge.

Problem 3: A uniform solid sphere

Find the electrostatic potential energy stored in a solid sphere of radius R with a total charge q distributed uniformly throughout its volume. Do this in the following three ways:

- (a) First, use what you know about the potential inside a uniformly charged sphere to evaluate

$$W = \frac{1}{2} \int_{\mathcal{V}} d\tau \rho(\vec{r}) V(\vec{r}), \quad (1)$$

where the integration volume \mathcal{V} is just the sphere itself.

- (b) Second, repeat the calculation from class where we integrated $|\vec{E}|^2$ over all space

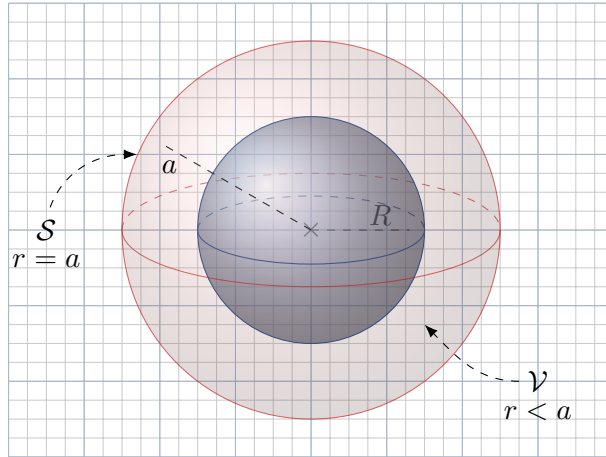
$$W = \frac{\epsilon_0}{2} \int_{\text{A.S.}} d\tau |\vec{E}(\vec{r})|^2. \quad (2)$$

I understand that you've already seen this worked out, but it's important to go through the details yourself.

- (c) Finally, evaluate the "intermediate" expression

$$W = \frac{\epsilon_0}{2} \int_{\mathcal{V}} d\tau |\vec{E}(\vec{r})|^2 + \frac{\epsilon_0}{2} \oint_{\mathcal{S}} d\vec{a} \cdot \vec{E}(\vec{r}) V(\vec{r}), \quad (3)$$

using the region inside a sphere of radius $a > R$ as the volume \mathcal{V} (so \mathcal{S} is the sphere $r = a$). Remember that the electric field in the region \mathcal{V} takes one form for $r \leq R$, and another for $R \leq r \leq a$.



You should of course get the same result in all three cases. In the last case, what happens to the two contributions (the surface integral and volume integral) as $a \rightarrow \infty$?

Problem 4: Concentric spherical shells

Consider two uniformly charged, concentric spherical shells. The inner shell, with radius R_i , carries a net charge $-q$. The outer shell, with radius R_o , carries a net charge q . Calculate the energy of this configuration in the following ways:

- Use formula (2) from the previous problem.
- Evaluate the total work needed to assemble this configuration

$$W_{\text{tot}} = W_i + W_o + \epsilon_0 \int_{\text{A.S.}} d\tau \vec{E}_i \cdot \vec{E}_o . \quad (4)$$

where \vec{E}_i and \vec{E}_o are the electric fields produced by the inner and outer spheres, respectively. The last term in (4) is the ‘interaction energy’. Can you give a physical interpretation of this quantity? How would you explain it to someone?

Problem 5: Similarities between Coulomb’s Law and Newtonian Gravity

Coulomb’s Law and Newton’s Universal Law of Gravitation both describe forces that vary with distance as $1/r^2$. As a result, the electric and gravitational fields have some properties in common with each other. **This problem is a little outside the scope of the class, but it’s an interesting exercise. It will be checked for completion, but it won’t be one of the graded problems.**

- Consider a hollow, uniform sphere with mass M and radius R . Show that its gravitational field at a distance r from the center is the same as that of a point mass when $r > R$, but zero when $r < R$. That is, evaluate the right-hand-side of

$$\vec{g} = -G \frac{M}{4\pi R^2} \oint_S da' \frac{\hat{z}}{r'^2} , \quad (5)$$

where the integral is over the surface of the sphere, and the factor $M/4\pi R^2$ is the (surface) mass density of the shell. Recall that the electric field of a hollow uniform sphere is zero inside, and resembles that of a point charge outside. Since Coulomb’s Law and Newtonian gravity have similar forms, it isn’t surprising that we get the same result. This isn’t a special property of electrostatics or gravity – it’s a property of any $1/r^2$ force.

- (b) Of course, if you were calculating the electric field for something as symmetric as a uniform charged shell you wouldn't bother evaluating the integral – you'd just use Gauss's Law! It must be that there is a Gauss's Law for gravity, too. Consider the integral formulas for the gravitational and electric fields

$$\vec{g} = -G \int_{\mathcal{V}} d\tau' \rho_m(\vec{r}') \frac{\hat{z}}{z^2} \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \rho(\vec{r}') \frac{\hat{z}}{z^2}, \quad (6)$$

where G is Newton's Constant, ρ_m is mass density, and (as usual) ρ is charge density. How would you write the integral and differential forms of Gauss's Law for gravity?

- (c) Would we get the same result in part (a) if Newtonian gravity (or Coulomb's Law) involved a $1/r$ force instead of a $1/r^2$ force? What do you obtain for \vec{g} in that case? You may use MATHEMATICA or similar tools for the integrals in this part.