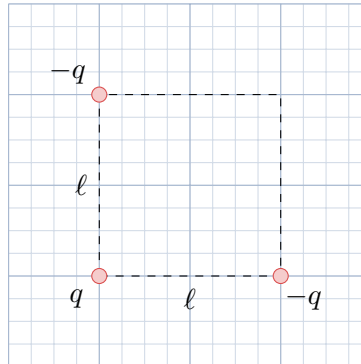


Homework 5: Electrostatic Potential Energy

Due: Monday, September 30

Problem 1: Point charges at the corners of a square

Three charges are situated at the corners of a square with sides of length ℓ , as shown in the figure below. How much work must one do to bring a fourth charge $+q$ in from infinitely far away and place it on the unoccupied corner, if the other three charges are already in place? How much work is required to bring all four charges together from infinitely far away?



Problem 2: A solid sphere with linear charge density

A solid sphere of radius R has a volume charge density $\rho(r) = \alpha r/R$ throughout its interior, where α is a constant with appropriate units. Find the potential energy of this distribution of charge (the work required to assemble it).

Problem 3: A uniform solid sphere

In class we integrated the square of the electric field over all space and found that the electrostatic potential energy of a total charge q distributed uniformly throughout the volume of a solid sphere of radius R is

$$W = \frac{q^2}{8\pi\epsilon_0} \frac{6}{5R} . \quad (1)$$

Reproduce this result in the following two ways:

- (a) First, use what you know about the potential inside a uniformly charged sphere to evaluate

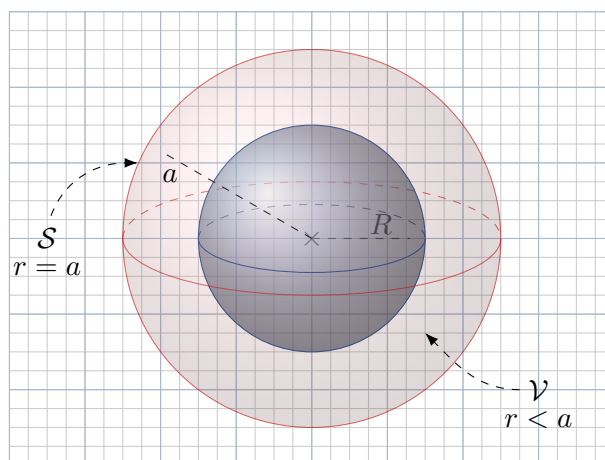
$$W = \frac{1}{2} \int_{\mathcal{V}} d\tau \rho(\vec{r}) V(\vec{r}) , \quad (2)$$

where the integration volume \mathcal{V} is just the sphere itself.

- (b) Evaluate the “intermediate” expression we obtained in class

$$W = \frac{\epsilon_0}{2} \int_{\mathcal{V}} d\tau |\vec{E}(\vec{r})|^2 + \frac{\epsilon_0}{2} \oint_{\mathcal{S}} d\vec{a} \cdot \vec{E}(\vec{r}) V(\vec{r}) , \quad (3)$$

using the region inside a sphere of radius $a > R$ as the volume \mathcal{V} (so \mathcal{S} is the sphere $r = a$). Remember that the electric field in the region \mathcal{V} takes one form for $0 \leq r \leq R$, and another for $R \leq r \leq a$.



You should of course get the same result in both cases. In the last case, what happens to the two contributions (the surface integral and volume integral) as $a \rightarrow \infty$?

Problem 4: Charged shells and the hydrogen atom

Consider two uniformly charged, concentric spherical shells. The inner shell has radius R_i and carries a net charge Q . The outer shell has radius R_o and carries a net charge $-Q$.

- (a) The total electric field \vec{E}_{tot} produced by the two shells is zero for $r < R_i$ and $r > R_o$, by spherical symmetry and Gauss's Law (since q_{enc} is zero in both cases). Between the two shells the total electric field is the field produced by the inner shell (since a Gaussian surface of radius $R_i < r < R_o$ encloses the inner shell's charge, but not the outer shell's charge). Calculate the total electrostatic potential energy for this configuration of charge by evaluating the integral of $|\vec{E}_{\text{tot}}|^2$ over all space, and show that it is equal to

$$W = \frac{\epsilon_0}{2} \int_{\text{A.S.}} d\tau \vec{E}_{\text{tot}} \cdot \vec{E}_{\text{tot}} = \frac{Q^2}{8\pi\epsilon_0} \left(\frac{1}{R_i} - \frac{1}{R_o} \right). \quad (4)$$

- (b) Let's use this result to make a simple model of a hydrogen atom. In your Modern Physics class you learned that, in its ground state, the H atom's electron resides in the spherically symmetric $1s$ orbital. The behavior of the electron is quantum mechanical and we can't assign it an exact radial position. However, if we look for it at different distances from the proton the most probable result in the $1s$ state is the "Bohr radius" $R_{\text{Bohr}} = 5.29 \times 10^{-11}$ m. So we'll model the $1s$ state as an inner shell with charge $e = 1.6 \times 10^{-19}$ C and an outer shell with charge $-e$ and radius R_{Bohr} . Ionizing the hydrogen atom means removing the electron so that it is very far away from the proton. In that case R_o for the outer shell goes to infinity and $1/R_o \rightarrow 0$. Then the difference in energy between an ionized Hydrogen atom and the $1s$ state should be

$$W_{\text{ion}} - W_{1s} = \frac{e^2}{8\pi\epsilon_0} \frac{1}{R_{\text{Bohr}}}. \quad (5)$$

This is called the "ionization energy" of the hydrogen atom in its ground state. Evaluate this using the values given above and $\epsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}$, then convert your answer from Joules to electron volts. How close is your answer to the accepted value?

***Problem 5: Similarities between Coulomb's Law and Newtonian Gravity**

Coulomb's Law and Newton's Universal Law of Gravitation both describe radial forces that vary with distance like $1/r^2$. As a result, the electric and gravitational fields have some properties in common with each other.

- (a) Consider a hollow, uniform sphere with mass M and radius R . Show that its gravitational field at a distance r from the center is the same as that of a point mass when $r > R$, but zero when $r < R$. That is, evaluate the right-hand-side of

$$\vec{g} = -G \frac{M}{4\pi R^2} \oint_S da' \frac{\hat{z}}{z^2}, \quad (6)$$

where the integral is over the surface of the sphere, and the factor $M/4\pi R^2$ is the surface (mass) density of the shell. Recall that the electric field of a hollow uniform sphere is zero inside, and resembles that of a point charge outside. Since Coulomb's Law and Newtonian gravity have similar forms, it isn't surprising that we get the same result. This isn't a special property of electrostatics or gravity – it's a property of any central force that depends on distance as $1/r^2$.

- (b) Of course, if you were calculating the electric field for something as symmetric as a uniform charged shell you wouldn't bother evaluating the integral – you'd just use Gauss's Law! It must be that there is a Gauss's Law for gravity, too. Consider the integral formulas for the gravitational and electric fields

$$\vec{g} = -G \int_V d\tau' \rho_m(\vec{r}') \frac{\hat{z}}{z^2} \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \rho(\vec{r}') \frac{\hat{z}}{z^2}, \quad (7)$$

where G is Newton's Constant, ρ_m is mass density, and (as usual) ρ is charge density. How would you write the integral and differential forms of Gauss's Law for gravity?

- (c) Would we get the same result in part (a) if Newtonian gravity (or Coulomb's Law) involved a $1/r$ force instead of a $1/r^2$ force? What do you obtain for \vec{g} in that case? You may use MATHEMATICA or similar tools for the integrals in this part.