Problem 1: Why We Love Conservative Forces
An object is moving in the x-y plane along the path \( y = x^2 + x - 3 \). It experiences a force
\[
\vec{F}(x, y) = \frac{2\alpha x}{(x^2 + y^2)^2} \hat{x} + \frac{2\alpha y}{(x^2 + y^2)^2} \hat{y},
\]
where \( \alpha \) is a constant. What is the work performed by this force as the object moves from the point \((2, 3)\) to the point \((0, -3)\) ?

**HINT:** The work is the integral of \( \vec{F} \cdot d\vec{r} \) along the path, from the start point to the end point. It will be messy if you try to do this directly. It can be done, but there is a much easier way to get the answer. Think about the title of the problem.

Problem 2: Divergence Theorem
The components of a vector function \( \vec{A} \) are given in cylindrical polar coordinates \( \{s, \phi, z\} \) as
\[
\vec{A} = \frac{s^2}{L} \sin \left( \frac{\pi z}{L} \right) \hat{s} + s \cos \left( \frac{\pi z}{L} \right) \hat{z},
\]
where \( L \) is a constant with units of length.

(a) Use a surface integral to calculate the flux of \( \vec{A} \) through the closed surface bounding the region \( 2L \leq s \leq 4L, 0 \leq z \leq L, 0 \leq \phi < 2\pi \).

(b) Evaluate the same integral using the divergence theorem.

Problem 3: Stokes’ Theorem
The components of a vector function \( \vec{V} \) are given in Cartesian coordinates as
\[
\vec{V} = \frac{x y}{a^2} \hat{x} + \left( \frac{x^2}{2a^2} + \frac{y z}{2a^2} \right) \hat{y} + \left( \frac{y^2}{4a^2} - \frac{2z}{a} \right) \hat{z},
\]
where \( a \) is a constant. Use Stokes’ theorem to calculate the integral
\[
\oint_P \vec{V} \cdot d\vec{r}
\]
where \( P \) is the rectangle \( ABCD \) with corners \( A = (-1, 1, 0), B = (4, 1, 0), C = (4, 4, 0), \) and \( D = (-1, 4, 0) \).

Problem 4: The Three-Dimensional Delta Function
Perform the following integrals involving the three-dimensional delta function:
(a) \[
\int_{\text{all space}} d\tau \left( r^2 - 5 \vec{r} \cdot \vec{a} + 2 a^2 \right) \delta^3(\vec{r} - \vec{a}),
\]
where \(\vec{r}\) is the position vector with magnitude \(r\), and \(\vec{a}\) is a constant vector with magnitude \(a\).

(b) \[
\int_{\mathcal{V}} d\tau |\vec{b} - \vec{r}|^2 \delta^3(5\vec{r}),
\]
where \(\mathcal{V}\) is a cube with sides of length 10, centered on the origin, and \(\vec{b} = 7\hat{x} - 24\hat{z}\).

**Problem 5: The Helmholtz Theorem**
Consider the following vector functions:

\[
\vec{F}_1 = y\hat{x} + z\hat{y} + 0\hat{z}\quad (5)
\]
\[
\vec{F}_2 = x^2 \hat{x} - y^2 \hat{y} + z^2 \hat{z}\quad (6)
\]
\[
\vec{F}_3 = -y z \hat{x} - z x \hat{y} - x y \hat{z}\quad (7)
\]

(a) Calculate the divergence and curl of \(\vec{F}_1\) and \(\vec{F}_2\).

(b) Which one of \(\vec{F}_1\) and \(\vec{F}_2\) can be written as the gradient of a scalar function? Find any scalar potential that does the job.

(c) Which one of \(\vec{F}_1\) and \(\vec{F}_2\) can be written as the curl of a vector function? Find any suitable vector potential.

(d) Show that \(\vec{F}_3\) can be written both as the gradient of a scalar function and as the curl of a vector function.

(e) Find a scalar potential and a vector potential for \(\vec{F}_3\).