Homework 2: More Vector Analysis

Due: Wednesday, September 5th

Important: You may not use MATHEMATICA or similar software on this homework. In fact, on most homework assignments you should assume that MATHEMATICA is not allowed. It is really important that you learn how to handle the math on these assignments without resorting to computers. There is a lot of understanding to be uncovered in working out the details for yourself.

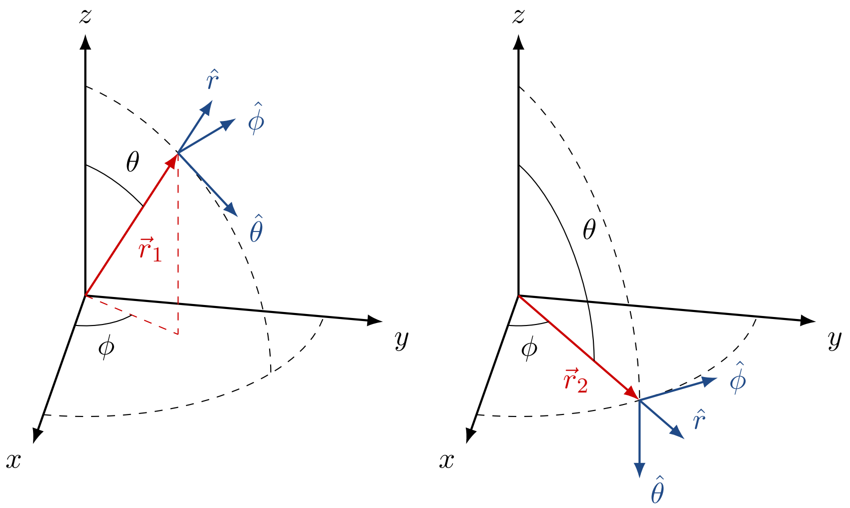
Problem 1: Spherical Polar Coordinates

Cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ) are related by

$$x = r\,\sin\theta\,\cos\phi \qquad y = r\,\sin\theta\,\sin\phi \qquad z = r\,\cos\theta\;. \tag{1}$$

Here $r \geq 0$ is the distance from the origin, $0 \leq \theta \leq \pi$ is the angle down from the z-axis, and $0 \leq \phi < 2\pi$ is the counterclockwise angle from the x-axis in the x-y plane. Notice that $\phi = 2\pi$ means the same thing as $\phi = 0$.

In Cartesian coordinates, the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ point in the directions of increasing x, y, and z, respectively. The same is true for the unit vectors in spherical polar coordinates: $\hat{r}$ points in the direction of increasing r, $\hat{\theta}$ in the direction of increasing θ, etc. In the figures below, you can see that at different points the directions associated with $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ change relative to $\hat{x}$, $\hat{y}$, and $\hat{z}$.



Let's review the relationship between unit vectors, vector components, and multi-variable derivatives like the gradient and divergence in the two coordinate systems. You may use results for general orthogonal coordinate systems that you learned in Math Methods, but you must derive the scale factors (show your work!) before using them.

(a) Derive expressions for the unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ as combinations of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. Give the coefficients as functions of r, θ, and ϕ.

(b) Now invert your results from part (a) and write $\hat{x}$, $\hat{y}$, and $\hat{z}$ as combinations of $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$. As before, give the coefficients as functions of r, θ, and ϕ.

(c) Derive expressions for the components of a vector in spherical polar coordinates, in terms of its Cartesian components. That is, starting from $\vec{V} = V_x\,\hat{x} + V_y\,\hat{y} + V_z\,\hat{z}$, use your results from part (b) to write the vector in the form $\vec{V} = V_r\,\hat{r} + V_\theta\,\hat{\theta} + V_\phi\,\hat{\phi}$, expressing V_r, V_θ, and V_ϕ as appropriate combinations of the Cartesian components V_x, V_y, and V_z.

(d) Derive Griffiths' equation (1.70) for the gradient of a function in spherical polar coordinates.

(e) Derive Griffiths' equation (1.71) for the divergence of a vector in spherical polar coordinates.

Problem 2: Why We Love Conservative Forces

An object is moving in the x-y plane along the path $y = x^2 + x - 3$. It experiences a force

$$\vec{F}(x,y) = \frac{2\,\alpha\,x}{(x^2+y^2)^2}\,\hat{x} + \frac{2\,\alpha\,y}{(x^2+y^2)^2}\,\hat{y}\ , \tag{2}$$

where α is a constant. What is the work performed by this force as the object moves from the point $(2, 3)$ to the point $(4, 17)$?

Hint: This problem is very, very messy if you try to integrate $\vec{F}\cdot d\vec{\ell}$ directly. There is a much easier way – think about the title of the problem.

Problem 3: Divergence Theorem

The components of a vector function $\vec{A}$ are given in cylindrical polar coordinates $\{s, \phi, z\}$ as

$$\vec{A} = \frac{s^3}{L}\,\sin\left(\pi\,\frac{z}{L}\right)\,\hat{s} + \frac{s^2}{\pi}\,\cos\left(\pi\,\frac{z}{L}\right)\,\hat{z}\ , \tag{3}$$

where L is a constant with units of length.

(a) Use a surface integral to calculate the flux of $\vec{A}$ through the closed surface bounding the region $L \le s \le 3\,L$, $0 \le z \le L$, $0 \le \phi < 2\pi$.

(b) Evaluate the same integral using the divergence theorem.

Problem 4: Stokes' Theorem

The components of a vector function $\vec{V}$ are given in Cartesian coordinates as

$$\vec{V} = -\frac{y^2}{2\,a^2}\,\hat{x} + \left(\frac{y\,z^2}{2\,a^3} - \frac{x\,y}{a^2}\right)\hat{y} + \left(\frac{y^2\,z}{2\,a^3} + 1\right)\hat{z}\ , \tag{4}$$

where a is a constant. Use Stokes' theorem to calculate the integral

$$\oint_P \vec{V}\cdot \mathrm{d}\vec{\ell} \tag{5}$$

where P is the rectangle $ABCD$ with corners $A = (-1, 2, 0)$, $B = (3, 2, 0)$, $C = (3, 5, 0)$, and $D = (-1, 5, 0)$.

Problem 5: The Three-Dimensional Dirac Delta

Perform the following integrals involving the three-dimensional Dirac delta.

(a) Let $\vec{a}$ be a constant vector with magnitude $|\vec{a}| = a$

$$\int_{\substack{\text{all}\\ \text{space}}} (r^2 - \vec{r}\cdot\vec{a} + a^2)\ \delta^3(\vec{r} - \vec{a}\,)\ d\tau\ .$$

(b) Let $\mathcal{V}$ be a cube with sides of length 3, centered on the origin, and $\vec{b} = -3\,\hat{x} + 4\,\hat{z}$

$$\int_{\mathcal{V}} \left|\,\vec{b} - \vec{r}\,\right|^2\, \delta^3(2\,\vec{r}\,)\; d\tau\ .$$

Problem 6: The Helmholtz Theorem

Consider the following vector functions:

$$\vec{F}_1 = y\,\hat{x} + z\,\hat{y} + 0\,\hat{z} \tag{6}$$

$$\vec{F}_2 = x^2\,\hat{x} - y^2\,\hat{y} + z^2\,\hat{z} \tag{7}$$

$$\vec{F}_3 = -y\,z\,\hat{x} - z\,x\,\hat{y} - x\,y\,\hat{z} \tag{8}$$

(a) Calculate the divergence and curl of $\vec{F}_1$ and $\vec{F}_2$.

(b) Which one of $\vec{F}_1$ and $\vec{F}_2$ can be written as the gradient of a scalar function? Find a scalar potential that does the job.

(c) Which one of $\vec{F}_1$ and $\vec{F}_2$ can be written as the curl of a vector function? Find a suitable vector potential.

(d) Show that $\vec{F}_3$ can be written both as the gradient of a scalar function and as the curl of a vector function.