▨ HELMHOLTZ THEORY OF VECTOR FIELDS

- Soon we will encounter differential equations for a vector field $\vec{V}$ that look like this:

$$\underbrace{\vec{\nabla}\cdot\vec{V}}_{\text{Div of }\vec{V}} = \underbrace{h}_{\text{Some scalar function}} \qquad \underbrace{\vec{\nabla}\times\vec{V}}_{\text{Curl of }\vec{V}} = \underbrace{\vec{W}}_{\text{Some vector function}}$$

- There are lots of sol'ns to these equations. For instance, if $\vec{V}$ is a sol'n then so is $\vec{V}$ plus any constant vector. If either h or $\vec{C}$ is zero, there are even more possibilities.

- You already knew this. If I ask you to solve

$$\frac{d f(x)}{dx} = -\frac{1}{x^2}$$

the answer is $f(x) = 1/x$ + any constant.

- To get a <u>unique</u> sol'n I must also give you a <u>boundary</u> <u>condition</u>. Something like:

$$\frac{d f(x)}{dx} = -\frac{1}{x^2} \quad \text{and} \quad f(1) = 5 \quad \Rightarrow \quad f(x) = \frac{1}{x} + 4$$

$$\frac{d f(x)}{dx} = -\frac{1}{x^2} \quad \text{and} \quad \lim_{x\to\infty} f(x) = 0 \quad \Rightarrow \quad f(x) = \frac{1}{x}$$

- The HELMHOLTZ THEOREM tells us that the problems we are interested in have <u>unique</u> sol'ns when we give sufficient boundary conditions on $\vec{V}$:

$$\vec{\nabla}\cdot\vec{V} = h, \quad \vec{\nabla}\times\vec{V} = \vec{W}, \quad \text{Bndy Cond on } \vec{V} \Rightarrow \text{Unique sol'n } \vec{V}$$

- It will often be the case that these equations are a bit simpler b/c either h or $\vec{W}$ is <u>zero</u>. In those cases, $\vec{V}$ has special properties that we might take advantage of.

- For instance, if $\vec{W} = 0$ everywhere — that is, if $\vec{\nabla} \times \vec{V} = 0$ at ever point in space, then we can always write $\vec{V}$ as the gradient of a scalar: $\vec{V} = -\vec{\nabla} U$.

- In fact, the following four conditions are equivalent. If any one of them is true for $\vec{V}$, then so are the other three:

  (i) $\vec{\nabla} \times \vec{V} = 0$ everywhere

  (ii) $\vec{V} = -\vec{\nabla} U \leftarrow$ Call U a 'SCALAR POTENTIAL'

  (iii) ${}_P\int_a^b d\vec{l} \cdot \vec{V}$ is independent of P, for any a & b

  (iv) $\oint_P d\vec{l} \cdot \vec{V} = 0$ for any and all closed paths P.

- Likewise, if $\vec{\nabla} \cdot \vec{V} = 0$ everywhere then we can always write $\vec{V}$ as the <u>curl</u> of some other vector $\vec{A}$ that we call a <u>VECTOR POTENTIAL</u>. (Often, it will be easier for us to find $\vec{A}$.)

- The following four conditions are all equivalent:

  (i) $\vec{\nabla} \cdot \vec{V} = 0$ everywhere

  (ii) $\vec{V} = \vec{\nabla} \times \vec{A}$

  (iii) $\oint_S d\vec{a} \cdot \vec{V} = 0$ for <u>any</u> closed S

  (iv) Given any closed path P, $\int_S d\vec{a} \cdot \vec{V}$ has the same value for <u>all</u> surfaces S w/ the same bndy P.

- In both cases the potential is not unique:
  - U & $U + \text{const}$ have the same gradient b/c $\vec{\nabla}(\text{constant}) = 0$
  - $\vec{A}$ & $\vec{A} + \vec{\nabla} g$ have same curl, b/c $\vec{\nabla} \times (\vec{\nabla} g) = 0$ for any scalar function g.

- More generally, if neither h nor $\vec{W}$ is zero, then the vector $\vec{V}$ can be written as the sum of a gradient and a curl:

$$\vec{V} = -\vec{\nabla} U + \vec{\nabla} \times \vec{A}$$

- There is more detail in Appendix B of the textbook. But for now, let me show you an example.

<u>Ex</u> Suppose I give you a scalar function $h(\vec{r})$ (remember, this is shorthand for $h(x,y,z)$) and ask you to find a vector field $\vec{F}$ such that $\vec{\nabla} \cdot \vec{F} = h$ and $\vec{\nabla} \times \vec{F} = 0$.

Since $\vec{\nabla} \times \vec{F} = 0$ I know that $\vec{F}$ can be written as the gradient of a scalar potential : $\vec{F} = -\vec{\nabla} U$.

<u>CLAIM</u>: One sol'n is $U(\vec{r}) = \frac{1}{4\pi} \int_{A.S.} d\tau' \frac{h(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$

<u>CHECK</u>: $\vec{\nabla} U(\vec{r}) = \frac{1}{4\pi} \int_{A.S.} d\tau' \; h(\underbrace{\vec{r}\,'}_{x',y',z'}) \; \vec{\nabla} \left(\underbrace{\frac{1}{|\vec{r} - \vec{r}\,'|}} \right)$

$\hat{x} \frac{d}{dx} + \hat{y} \frac{d}{dy} + \hat{z} \frac{d}{dz}$

$dx'dy'dz'$

Integrate over <u>all space</u>

All the x,y,z dependence is in here; d/dx etc don't care about x', y', z'

$$\vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) = \vec{\nabla} \left(\left((x-x')^2 + (y-y')^2 + (z-z')^2 \right)^{-1/2} \right)$$

$$= -\frac{1}{2} \left((x-x')^2 + (y-y')^2 + (z-z')^2 \right)^{-3/2} 2(x-x') \hat{x} + (\dots) \hat{y} + (\dots) \hat{z}$$

$$= -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\rightarrow \vec{\nabla} U = -\frac{1}{4\pi}\int_{A.S.} d\tau'\, h(\vec{r}\,')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\rightarrow \vec{F} = -\vec{\nabla} U = \frac{1}{4\pi}\int_{A.S.} d\tau'\, h(\vec{r}\,')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Again, $\vec{\nabla}$ only cares about x, y, z dependence in the integrand.

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{4\pi}\int_{A.S.} d\tau'\, h(\vec{r}\,')\, \underbrace{\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}$$

$$\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{r}\,' - \vec{r})$$

$$= \cancel{\frac{1}{4\pi}}\int_{A.S.} d\tau'\, h(\vec{r}\,')\, \cancel{4\pi}\,\delta^3(\vec{r}\,' - \vec{r})$$

$$\Rightarrow \vec{\nabla}\cdot\vec{F} = h(\vec{r}) \checkmark$$

- This example looks a bit abstract, though you may notice that it looks very much like the Coulomb integrals for the electric field that we wrote down in class. Don't worry, we'll develop these ideas in more detail as we need them.