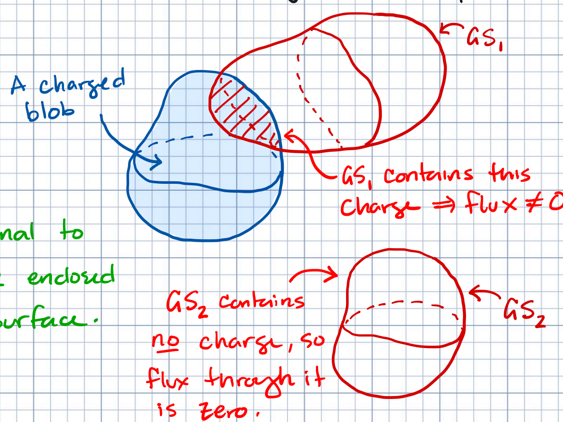
Gauss's Law for a Spherically Symmetric Distribution of Charge

- Let's look @ an example of how we can use Gauss's Law to determine the electric field of a very symmetric distribution of charge <u>without</u> setting up and evaluating Coulomb integrals.

- First, Gauss's Law is <u>always</u> <u>true</u>. It is a basic fact about electric fields & the charges that produce them:

$$\underbrace{\int_{GS} d\vec{a}\cdot\vec{E}} = \underbrace{\frac{q_{enc}}{\varepsilon_0}}$$

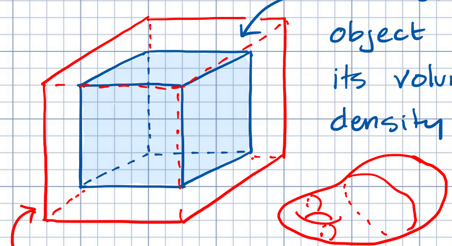
The flux of $\vec{E}$ across <u>any</u> closed surface GS is.... ... proportional to the charge enclosed by that surface.



- We often talk about charge being spread over an object or throughout its volume. So once we describe a charge distribution, you may have a particular surface in mind (i.e., a sphere, or a cube, etc).

- But usually that is <u>not</u> the surface we are talking about with Gauss's Law. Instead, we have some imaginary surface in mind. It may (as we'll see below) have a similar shape but a different size. Or it may be some entirely different shape.

- To avoid confusion, I will use $\mathcal{S}$ and $\mathcal{V}$ to denote a surface and volume associated with an actual charged object. And I will use GS and GV to describe a 'Gaussian surface' — a surface I am using in Gauss's Law — and the volume inside it.

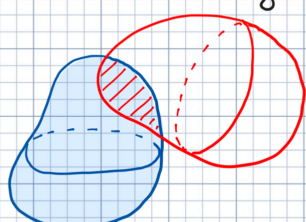


This cube is an actual physical object w/ charge density ρ in its volume $\mathcal{V}$ & surface charge density σ on its surface $\mathcal{S}$.

This Gaussian surface has a totally unrelated shape.

This cube is a Gaussian surface w/ the same shape as the object, but it's <u>bigger</u>.

- If I tell you about a distribution of charge, and then I describe a G.S. to you, it should be straightforward (in principle) for you to tell me how much of the charge is enclosed by the G.S.



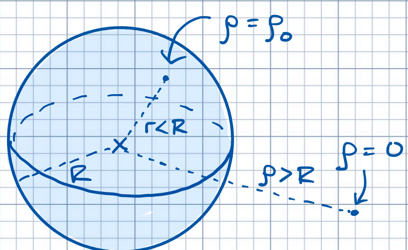
The q_{enc} for this G.S. is the charge in the part of the blue blop <u>inside</u> the G.S.

- Now, Gauss's Law can help us determine $\vec{E}$ when the distribution of charge producing the electric field is <u>very symmetric</u>.

- The procedure for finding $\vec{E}$ will involve Gaussian surfaces that have the same basic shape or symmetry as the charge. So let's look @ a simple example to see how it works.

- Suppose I show you a sphere w/ radius R that has a uniform charge density. We'll call the constant charge density ρ_0.

- This distribution of charge has <u>SPHERICAL SYMMETRY</u> because I can give you a complete description of the charge using only the distance r from a single point – the center of the sphere:



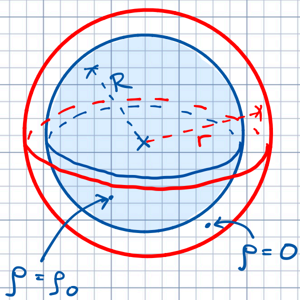
$$\rho(\vec{r}) = \begin{cases} \rho_0 \, , & r<R \\ 0 \, , & r>R \end{cases}$$

- What surfaces have spherical symmetry? Well, what surfaces can I describe using only the distance r from a central point? Spheres.

- So we'll use spheres w/ radius r & centered on the same point as our Gaussian surfaces.

- If $r > R$, our G.S. completely encloses the charged sphere. If $r < R$, the G.S. encloses only part of it.

The charge enclosed by a G.S. w/ $r > R$ is:



$$q_{enc} = \int_{GV} d\tau' \rho(\vec{r}')$$

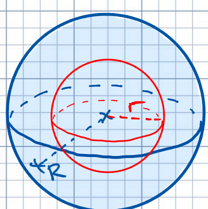
$$= \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \int_0^r dr' r'^2 \rho(\vec{r}')$$

$$= \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \Big(\underbrace{\int_0^R dr' r'^2 \rho_0}_{r' < R} + \underbrace{\int_R^r dr' \, 0}_{R < r' < r} \Big)$$

$$\hookrightarrow q_{enc} = \frac{4}{3}\pi R^3 \rho_0$$

- This makes sense. All the charge in our sphere — it's just $(4/3)\pi R^3$ times ρ_0 since the density is constant — is inside the G.S. in this case.

- If $r < R$, the G.S. encloses only <u>part</u> of the charge.



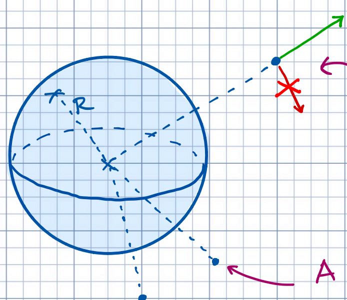
$$q_{enc} = \int_{GV} d\tau' \rho(\vec{r}')$$

$$= \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \int_0^r dr' r'^2 \underbrace{\rho(\vec{r}')}_{\rho_0 \text{ everywhere in this GV}}$$

$$= \frac{4}{3}\pi r^3 \rho_0$$

Since this G.S. is smaller than the charged sphere it captures only a fraction of its charge.

- So how does this let us determine $\vec{E}$?
- First, we expect that the symmetries of the charge distribution tell us about $\vec{E}$.
- In this case, if we use SPC w/ the origin @ the center of the sphere, it seems like $\vec{E}$ could depend on r, but <u>not</u> θ or ϕ. We expect $\vec{E}$ to get bigger or smaller as we move toward or away from the sphere. But it looks the same from all directions (spherical symmetry!) so θ & ϕ can't be relevant.
- Likewise, symmetry suggests $\vec{E}$ could point in the $\hat{r}$ direction, but not $\hat{\theta}$ or $\hat{\phi}$.



By symmetry, contributions to $\vec{E}$ in $\hat{\theta}$ & $\hat{\phi}$ dir. will all cancel out. Net result is $\vec{E}$ in $\hat{r}$ direction.

A spherically symm. charge distribution looks the same from two pts. w/ same r & different θ, ϕ.

- So a spherically symmetric charge distribution (w/ $r=0$ @ its center) implies an electric field of the form:

$$\vec{E}(\vec{r}) = E(r)\,\hat{r}$$

- Now recall Gauss's Law!

$$\int_{G.S.} d\vec{a} \cdot \hat{r}\, E(r) = \frac{q_{enc}}{\varepsilon_0}$$

- Suppose the G.S. is a sphere centered @ $r=0$ (the center of the charged sphere). Then:

$$\int_{GS} d\vec{a} \cdot \hat{r}\, E(r) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, r^2\, \overbrace{\hat{r} \cdot \hat{r}}^{1}\, E(r)$$

$$= 4\pi r^2 E(r)$$

We don't <u>know</u> $E(r)$ yet, but this doesn't stop us from evaluating the integral b/c every point on our G.S. has the <u>same</u> value of r. It's a sphere!

- If the radius of the G.S. is $r > R$, then $q_{enc} = \frac{4}{3}\pi R^3 \rho_0$:

$$\underbrace{4\pi r^2 E(r)}_{\text{Flux of spherically symm. } \vec{E} \text{ through a spherical G.S.}} = \frac{1}{\varepsilon_0} \underbrace{\frac{4}{3}\pi R^3 \rho_0}_{\text{Charge enclosed by that G.S. when } r>R.}$$

$$\Rightarrow E(r>R) = \frac{1}{4\pi\varepsilon_0} \frac{4}{3}\pi R^3 \rho_0 \frac{1}{r^2}$$

$$\Rightarrow \vec{E}(r>R) = \frac{1}{4\pi\varepsilon_0} \frac{4}{3}\pi R^3 \rho_0 \frac{\hat{r}}{r^2}$$

- In other words, outside $(r>R)$ a spherically symm. distribution of charge, the electric field is the same as for a point charge.

- But what about $r<R$? When the G.S. is smaller than the charged sphere it encloses only <u>part</u> of its charge: $q_{enc} = \frac{4}{3}\pi r^3 \rho_0$

$$\hookrightarrow 4\pi \cancel{r^2} E(r) = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^{\cancel{3}} \rho_0$$

$$\Rightarrow E(r<R) = \frac{1}{4\pi\varepsilon_0}\frac{4}{3}\pi r \rho_0$$

This has the same <u>units</u> as $E(r>R)$, but instead of a factor of R^3/r^2 it has a factor of r. We could also write it as:

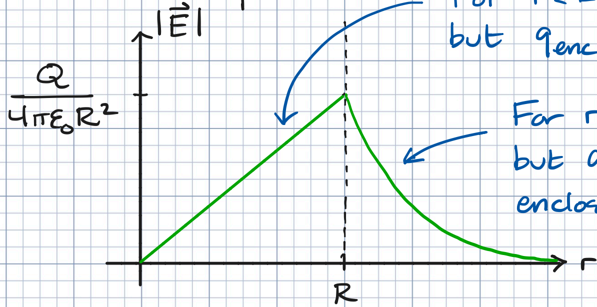
$$E(r<R) = \frac{1}{4\pi\varepsilon_0}\frac{4}{3}\pi R^3 \rho_0 \frac{r}{R^3}$$

$$\Rightarrow \vec{E}(r<R) = \frac{1}{4\pi\varepsilon_0}\frac{4}{3}\pi R^3 \rho_0 \frac{r}{R^3}\hat{r}$$

- So <u>inside</u> the uniform charged sphere, the electric field grows <u>linearly</u>. At $r=R$, the two expressions (for $r>R$ & $r<R$) agree.
- If we use $Q = \frac{4}{3}\pi R^3 \rho_0$, then $\vec{E}$ is:

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\dfrac{r}{R^3}\hat{r}, & r \le R \\[2ex] \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\hat{r}}{R^2}, & r \geq R \end{cases}$$

- Here's a plot:



For $r < R$, the flux is $\propto r^2 E(r)$, but $q_{enc} \propto r^3$, so $E(r) \propto r$.

For $r > R$, the flux is $\propto r^2 E(r)$, but q_{enc} is constant (any $r > R$ encloses the entire charged sphere!) so $E(r) \propto \frac{1}{r^2}$.

- In this example, our expressions for q_{enc} were specific to a uniform ($\rho =$ constant) charge density.
- But everything else we did (our assumptions about $\vec{E}$, our expressions for the flux) depended only on spherical symmetry.
- Therefore, if we had $\rho(r)$ rather than ρ_0 — which is still spherically symmetric — the analysis would be exactly the same except for the part where we calculate q_{enc}!
- Gauss's Law can help us out in three situations:
  1) Spherical symmetry: $\rho(\vec{r}) = \rho(r)$, where r is the distance from a central point ($r=0$).
  2) Cylindrical or axial symmetry: $\rho(\vec{r}) = \rho(s)$, where s is the distance from a central axis ($s=0$).
  3) Planar Symmetry: $\rho(\vec{r}) = \rho(z)$, where z is the distance from a central plane ($z=0$).

- Gauss's Law is <u>always</u> true, for any G.S. But those 3 situations are the only ones where we can exploit it to find $\vec{E}$ without having to set up & evaluate Coulomb integrals.
- Finally, to use Gauss's Law we have to be able to say something about the flux

$$\int_{GS} da\,\hat{n}\cdot\vec{E}$$

even though we <u>don't</u> <u>yet</u> <u>know</u> $\vec{E}$. The key is to identify what we <u>do</u> know about $\vec{E}$ and then pick the right surface.
- If we know that $\vec{E}$ depends on a single co-ordinate u & has direction $\hat{E}$:

$$\vec{E} = E(u)\,\hat{E}$$

then we look for a G.S. that has $u =$ constant, or is made up of multiple surfaces some of which have $u =$ constant & some of which have $\hat{n}\cdot\hat{E} = 0$.
- For example, with planar symmetry ($\rho = \rho(z)$) we expect $\vec{E} = E(z)\,\hat{z}$. So for our G.S. we might use 'boxes' w/ bottom @ $z = z_1$, top @ $z = z_2$, and sides where $\hat{n} = \pm\hat{x}$ or $\pm\hat{y}$ so $\hat{n}\cdot\hat{z} = 0$.