GAUSS’S LAW FOR A SPHERICALLY SYMMETRIC DISTRIBUTION OF CHARGE

- Let’s look @ an example of how we can use Gauss’s Law to determine the electric field of a very symmetric distribution of charge without setting up and evaluating Coulomb integrals.

- First, Gauss’s Law is always true. It is a basic fact about electric fields & the charges that produce them:

\[ \int \mathbf{d}a \cdot \mathbf{E} = \frac{q_{\text{enc}}}{\varepsilon_0} \]

The flux of \( \mathbf{E} \) across any closed surface \( GS \) is …

… proportional to the charge enclosed by that surface.

- We often talk about charge being spread over an object or throughout its volume. So once we describe a charge distribution, you may have a particular surface in mind (i.e., a sphere, or a cube, etc).

- But usually that is not the surface we are talking about with Gauss’s Law. Instead, we have some imaginary surface in mind. It may (as we’ll see below) have a similar shape but a
Different size. Or it may be some entirely different shape.

- To avoid confusion, I will use $S$ and $V$ to denote a surface and volume associated with an actual charged object. And I will use $\sigma S$ and $\rho V$ to describe a 'Gaussian surface' — a surface I am using in Gauss's Law — and the volume inside it.

\[ \text{This cube is an actual physical object with charge density } \rho \text{ in its volume } V \text{ and surface charge density } \sigma \text{ on its surface } S. \]

\[ \text{This cube is a Gaussian surface with the same shape as the object, but it's bigger.} \]

- If I tell you about a distribution of charge, and then I describe a G.S. to you, it should be straightforward (in principle) for you to tell me how much of the charge is enclosed by the G.S.

\[ \text{The qenc for this G.S. is the charge in the part of the blue blob inside the G.S.} \]
- Now, Gauss's Law can help us determine $\vec{E}$ when the distribution of charge producing the electric field is very symmetric.

- The procedure for finding $\vec{E}$ will involve Gaussian surfaces that have the same basic shape or symmetry as the charge. So let's look at a simple example to see how it works.

- Suppose I show you a sphere w/ radius $R$ that has a uniform charge density. We'll call the constant charge density $\rho_0$.

- This distribution of charge has **spherical symmetry** because I can give you a complete description of the charge using only the distance $r$ from a single point - the center of the sphere:

$$\rho(r) = \begin{cases} 
\rho_0, & r < R \\
0, & r \geq R 
\end{cases}$$

- What surfaces have spherical symmetry? Well, what surfaces can I describe using only the distance $r$ from a central point? Spheres.

- So we'll use spheres w/ radius $R$ & centered on the same point as our Gaussian surfaces.
- If \( r > R \), our G.S. completely encloses the charged sphere. If \( r < R \), the G.S. encloses only part of it.

The charge enclosed by a G.S. w/\( r > R \) is:

\[
q_{\text{enc}} = \int_{\Omega} dV' \rho(r')
\]

\[
= \int_{0}^{\pi} d\phi' \int_{0}^{\pi} d\theta' \sin \theta' \int_{0}^{R} dr' r'^2 \rho(r')
\]

\[
= \int_{0}^{\pi} d\theta' \sin \theta' \left( \frac{2}{3} \pi R^3 \rho_0 + \frac{1}{R} \int_{0}^{R} dr' r'^2 \rho(r') \right)
\]

\[
\iff R \frac{q_{\text{enc}}}{2} = \frac{4}{3} \pi R^3 \rho_0
\]

- This makes sense. All the charge in our sphere – it’s just \( (4/3)\pi R^3 \) times \( \rho_0 \) since the density is constant – is inside the G.S. in this case.

- If \( r < R \), the G.S. encloses only part of the charge.

\[
q_{\text{enc}} = \int_{\Omega} dV' \rho(r')
\]

\[
= \int_{0}^{\pi} d\phi' \int_{0}^{\pi} d\theta' \sin \theta' \int_{0}^{r} dr' r'^2 \rho(r')
\]

\[
= \frac{4}{3} \pi r^3 \rho_0
\]

Since this G.S. is smaller than the charged sphere, it captures only a fraction of its charge.

Once \( r > R \) it encloses all the charge.

Radius of G.S.
- So how does this let us determine \( \vec{E} \)?
- First, we expect that the symmetries of the charge distribution tell us about \( \vec{E} \).
- In this case, if we use SPC w/ the origin @ the center of the sphere, it seems like \( \vec{E} \) could depend on \( r \), but not \( \theta \) or \( \phi \). We expect \( \vec{E} \) to get bigger or smaller as we move toward or away from the sphere. But it looks the same from all directions (spherical symmetry!), so \( \theta \) & \( \phi \) can’t be relevant.
- Likewise, symmetry suggests \( \vec{E} \) could point in the \( \hat{r} \) direction, but not \( \hat{\theta} \) or \( \hat{\phi} \).

\[ \hat{E}(r) = E(r) \hat{r} \]
- Now recall Gauss's Law!

$$\int_{\text{G.S.}} d\mathbf{a} \cdot \hat{r} E(r) = \frac{q_{\text{enc}}}{\varepsilon_0}$$

- Suppose the G.S. is a sphere centered @ $r = 0$ (the center of the charged sphere). Then:

$$\int_{\text{G.S.}} d\mathbf{a} \cdot \hat{r} E(r) = \int_0^{2\pi} d\phi \int_0^{\pi} \frac{1}{r^2} r^2 \hat{r} \cdot \hat{r} E(r)$$

$$= 4\pi r^2 E(r)$$

We don't know $E(r)$ yet, but this doesn't stop us from evaluating the integral because every point on our G.S. has the same value of $r$. It's a sphere!

- If the radius of the G.S. is $r > R$, then $q_{\text{enc}} = \frac{4}{3} \pi R^3 \rho_0$.

$$4\pi r^2 E(r) = \frac{1}{\varepsilon_0} \frac{4}{3} \pi R^3 \rho_0$$

Flux of spherically symm. $E$ through a spherical G.S.

$$\Rightarrow E(r > R) = \frac{1}{4\pi \varepsilon_0} \frac{4}{3} \pi R^3 \rho_0 \frac{1}{r^2}$$

$$\Rightarrow \vec{E}(r > R) = \frac{1}{4\pi \varepsilon_0} \frac{4}{3} \pi R^3 \rho_0 \hat{r} \frac{1}{r^2}$$

Charge enclosed by that G.S. when $r > R$. 

<table>
<thead>
<tr>
<th>Flux of spherically symm. $E$ through a spherical G.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4\pi r^2 E(r) = \frac{1}{\varepsilon_0} \frac{4}{3} \pi R^3 \rho_0$</td>
</tr>
</tbody>
</table>

Charge enclosed by that G.S. when $r > R$. 

$$\Rightarrow E(r > R) = \frac{1}{4\pi \varepsilon_0} \frac{4}{3} \pi R^3 \rho_0 \frac{1}{r^2}$$

$$\Rightarrow \vec{E}(r > R) = \frac{1}{4\pi \varepsilon_0} \frac{4}{3} \pi R^3 \rho_0 \hat{r} \frac{1}{r^2}$$

<table>
<thead>
<tr>
<th>Charge enclosed by that G.S. when $r &gt; R.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{\text{enc}} = \frac{4}{3} \pi R^3 \rho_0$</td>
</tr>
</tbody>
</table>

$$\Rightarrow E(r > R) = \frac{1}{4\pi \varepsilon_0} \frac{4}{3} \pi R^3 \rho_0 \frac{1}{r^2}$$

$$\Rightarrow \vec{E}(r > R) = \frac{1}{4\pi \varepsilon_0} \frac{4}{3} \pi R^3 \rho_0 \hat{r} \frac{1}{r^2}$$

**Note:** The notation used in the text is consistent with standard physics notation, but some symbols and expressions may differ slightly from the original text due to the nature of handwriting and transcription.
- In other words, outside \((r > R)\) a spherically symm. distribution of charge, the electric field is the same as for a point charge.

- But what about \(r < R\)? When the G.S. is smaller than the charged sphere it encloses only part of its charge: \(q_{	ext{enc}} = \frac{4}{3} \pi r^3 \rho_0\)

\[
\Rightarrow 4 \pi r^2 E(r) = \frac{1}{\varepsilon_0} \frac{4}{3} \pi r^3 \rho_0
\]

\[
\Rightarrow E(r < R) = \frac{1}{4 \pi \varepsilon_0} \frac{4}{3} \pi r \rho_0
\]

This has the same units as \(E(r > R)\), but instead of a factor of \(R^3/r^2\) it has a factor of \(r\). We could also write it as:

\[
E(r < R) = \frac{1}{4 \pi \varepsilon_0} \frac{4}{3} \pi R^3 \rho_0 \frac{r}{R^3}
\]

\[
\Rightarrow \vec{E}(r < R) = \frac{1}{4 \pi \varepsilon_0} \frac{4}{3} \pi R^3 \rho_0 \frac{r}{R^3} \hat{r}
\]

- So inside the uniform charged sphere, the electric field grows linearly. At \(r = R\), the two expressions (for \(r > R\) \& \(r < R\)) agree.

- If we use \(Q = \frac{4}{3} \pi R^3 \rho_0\), then \(\vec{E}\) is:

\[
\vec{E}(r) = \begin{cases} 
\frac{Q}{4 \pi \varepsilon_0} \frac{r}{R^3} \hat{r}, & r \leq R \\
\frac{Q}{4 \pi \varepsilon_0} \frac{\hat{r}}{R^2}, & r > R
\end{cases}
\]
Here's a plot:

- For $r < R$, the flux is $\propto r^2 E(r)$, but $q_{enc} \propto r^3$, so $E(r) \propto r$.
- For $r > R$, the flux is $\propto r^2 E(r)$, but $q_{enc}$ is constant (any $r > R$ encloses the entire charged sphere!) so $E(r) \propto \frac{1}{r^2}$.

- In this example, our expressions for $q_{enc}$ were specific to a uniform ($\rho =$ constant) charge density.
- But everything else we did (our assumptions about $E$, our expressions for the flux) depended only on spherical symmetry.
- Therefore, if we had $\rho(r)$ rather than $\rho_0$ — which is still spherically symmetric — the analysis would be exactly the same except for the part where we calculate $q_{enc}$.
- Gauss's Law can help us out in three situations:

  1. Spherical symmetry: $\rho(\vec{r}) = \rho(r)$, where $r$ is the distance from a central point $(r=0)$.
  2. Cylindrical or axial symmetry: $\rho(\vec{r}) = \rho(s)$, where $s$ is the distance from a central axis $(s=0)$.
  3. Planar symmetry: $\rho(\vec{r}) = \rho(z)$, where $z$ is the distance from a central plane $(z=0)$. 
- Gauss's Law is always true, for any G.S. But those 3 situations are the only ones where we can exploit it to find \( \vec{E} \) without having to set up \( \vec{E} \); evaluate Coulomb integrals.

- Finally, to use Gauss's Law we have to be able to say something about the flux

\[
\int da \hat{n} \cdot \vec{E}
\]

even though we don't yet know \( \vec{E} \). The key is to identify what we do know about \( \vec{E} \) and then pick the right surface.

- If we know that \( \vec{E} \) depends on a single coordinate \( u \), \( \vec{E} \) has direction \( \vec{E} = E(u) \hat{E} \)

then we look for a G.S. that has \( u = \text{constant} \), or is made up of multiple surfaces some of which have \( u = \text{constant} \) \( \vec{E} \), some of which have \( \hat{n} \cdot \vec{E} = 0 \).

- For example, with planar symmetry \( (p=p(z)) \) we expect \( \vec{E} = E(z) \hat{z} \). So for our G.S. we might use 'boxes' w/ bottom @ \( z = z_1 \), top @ \( z = z_2 \), and sides where \( \hat{n} = \pm \hat{x} \text{ or } \pm \hat{y} \) so \( \hat{n} \cdot \vec{E} = 0 \).