MAXWELL:

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

- Since $\vec{\nabla}\cdot\vec{B}=0$ everywhere, $\exists\ \vec{A}$ st $\vec{B}=\vec{\nabla}\times\vec{A}$.
- But $\vec{\nabla}\times\vec{E}\neq 0$, so $\vec{E}$ isn't necessarily a gradient.

$$\vec{E} = -\vec{\nabla}V + \vec{\mathcal{F}} \to -\overbrace{\cancel{\vec{\nabla}\times\vec{\nabla}}}^{0}V + \vec{\nabla}\times\vec{\mathcal{F}} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}$$
$$\to \vec{\nabla}\times\vec{\mathcal{F}} = \vec{\nabla}\times\left(-\frac{\partial\vec{A}}{\partial t}\right)$$

So $\mathcal{F} = -\partial\vec{A}/\partial t$ plus some gradient which just gets absorbed into V.

$$\Rightarrow \boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial}{\partial t}\vec{A} \qquad \vec{B} = \vec{\nabla}\times\vec{A}}$$

GAUGE TRANSFORMATIONS

- Since $\vec{B}=\vec{\nabla}\times\vec{A}$, $\vec{B}$ invariant under $\vec{A}\to\vec{A}+\vec{\nabla}\lambda$
- Suppose $\vec{A}\to\vec{A}+\vec{\nabla}\lambda$ & $V\to V+\chi$. Then

$$\vec{E} = -\vec{\nabla}V - \frac{\partial}{\partial t}\vec{A} \to \vec{E} - \vec{\nabla}\chi - \frac{\partial}{\partial t}\vec{\nabla}\lambda = \vec{E} - \vec{\nabla}\left(\chi + \frac{\partial\lambda}{\partial t}\right)$$

So $\vec{E}$ invariant if $\chi = -\partial\lambda/\partial t$

- Both $\vec{E}$ & $\vec{B}$ invariant under $\boxed{V\to V-\frac{\partial\lambda}{\partial t},\ \vec{A}\to\vec{A}+\vec{\nabla}\lambda}$ for arbitrary function $\lambda(\vec{r},t)$.
- This includes the gauge transf. $V\to V+\text{const.}$ & $\vec{A}\to\vec{A}+\vec{\nabla}f(\vec{r})$ of E/S & M/S.

MAXWELL w/ POTENTIALS

(i) $\vec{\nabla}\cdot\vec{E} = -\nabla^2 V - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \rho/\varepsilon_0$

$$\Rightarrow \nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\varepsilon_0$$

(ii) $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ ✓

(iii) $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(-\vec{\nabla}V - \frac{\partial}{\partial t}\vec{A}\right) = -\overbrace{\cancel{\vec{\nabla}\times\vec{\nabla}}}^{0}V - \frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} = -\frac{\partial\vec{B}}{\partial t}$ ✓

(iv) $\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} - \mu_0\varepsilon_0\vec{\nabla}\frac{\partial V}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$

So the two non-trivial eqns are:

$$\boxed{\nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\varepsilon_0}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} - \mu_0\varepsilon_0\vec{\nabla}\frac{\partial V}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\hookrightarrow \boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \vec{\nabla}\left(\frac{1}{c^2}\frac{\partial V}{\partial t} + \vec{\nabla}\cdot\vec{A}\right)}$$

COULOMB GAUGE

- Can always choose $\lambda(\vec{r},t)$ st $\vec{\nabla}\cdot\vec{A} = 0$.
- Suppose $\vec{\nabla}\cdot\vec{A}_0 \neq 0$. Let $\vec{A} = \vec{A}_0 + \vec{\nabla}\lambda$, $V = V_0 - \frac{\partial\lambda}{\partial t}$. Then:

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{A}_0 + \nabla^2\lambda$$

$$\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0$$

Since we can solve this for λ, we can always find a gauge in which the potentials satisfy $\vec{\nabla}\cdot\vec{A} = 0$.

- In Coulomb gauge, the eqns. for the potentials are:

$$\boxed{\begin{aligned} &\nabla^2 V = -\rho/\varepsilon_0 \\ &\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\vec{\nabla}\left(\frac{\partial V}{\partial t}\right) \end{aligned}}$$

So $V(\vec{r},t)$ is <u>very</u> simple to calculate in Coulomb gauge, but $\vec{A}(\vec{r},t)$ is <u>not</u>.

LORENTZ GAUGE

- Likewise, we can find $\lambda(\vec{r},t)$ to get potentials that satisfy

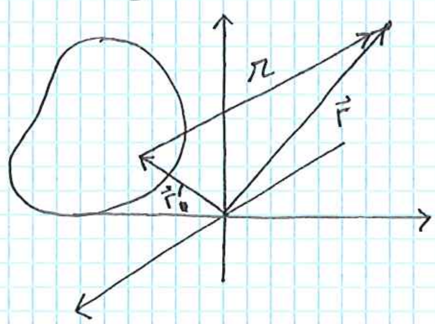
$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$$

- This is Lorentz gauge, where the eqns for $\vec{A}$ & V are both <u>wave equations</u> w/ sources:

$$\boxed{\begin{aligned} &\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon_0 \\ &\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \end{aligned}}$$

RETARDED POTENTIALS

- The eqns for $\vec{A}$ & V in Lorentz gauge show that changes in the potentials propagate @ speed c. So the potential @ pos. $\vec{r}$ & time t is influenced by the charge & current densities @ <u>earlier</u> times, depending on the location of those sources.



← At time t, the pt. $\vec{r}$ receives info about ρ @ $\vec{r}'$ as it was not @ t, but @ an earlier time $t_{\mathscr{r}}$. This info travels a distance $\mathscr{r} = |\vec{r}-\vec{r}'|$ @ constant speed c, so

$$\underbrace{\Delta t}_{t - t_{\mathscr{r}}} = \frac{|\vec{r}-\vec{r}'|}{c}$$

$$\Rightarrow \boxed{t_{\mathscr{r}} = t - \frac{\mathscr{r}}{c}}$$

← 'RETARDED TIME'

- So to compute V & $\vec{A}$, we need to sum up contributions from ρ & $\vec{J}$ @ points $\vec{r}'$, @ the correct retarted time $t_{\mathscr{r}}$ according to their distance from $\vec{r}$:

$$\boxed{\begin{aligned} V(\vec{r},t) &= \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\rho(\vec{r}',t_{\mathscr{r}})}{\mathscr{r}} \\ \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{J}(\vec{r}',t_{\mathscr{r}})}{\mathscr{r}} \end{aligned}}$$

← LORETZ GAUGE

- Do these solve the Lorentz gauge eqns?

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \overset{?}{=} -\rho/\varepsilon_0$$

$$\vec{\nabla} V = \frac{1}{4\pi\varepsilon_0}\int d\tau' \left(\rho(\vec{r}',t_{\mathscr{r}})\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\vec{\nabla}\rho(\vec{r}',t_{\mathscr{r}})\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\int d\tau' \left(-\rho(\vec{r}',t_{\mathscr{r}})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{1}{\mathscr{r}}\frac{\partial\rho(\vec{r}',t_{\mathscr{r}})}{\partial t_{\mathscr{r}}}\underbrace{\vec{\nabla} t_{\mathscr{r}}}_{-\frac{1}{c}\vec{\nabla}\mathscr{r} = -\frac{1}{c}\hat{\mathscr{r}}}\right)$$

$$\vec{\nabla} V = \frac{1}{4\pi\varepsilon_0}\int d\tau' \left(-\rho(\vec{r}', t_r)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{1}{\mathscr{r}}\frac{\partial \rho(\vec{r}', t_r)}{\partial t_r}\left(-\frac{1}{c}\hat{\mathscr{r}}\right)\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\int d\tau' \left(-\rho(\vec{r}', t_r)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} - \frac{1}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}}\frac{\partial \rho(\vec{r}', t_r)}{\partial t_r}\right)$$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int d\tau' \left(-\vec{\nabla}\rho(\vec{r}', t_r)\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} - \rho(\vec{r}', t_r)\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} - \frac{1}{c}\left(\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)\frac{\partial \rho(\vec{r}, t_r)}{\partial t_r} - \frac{1}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot\vec{\nabla}\frac{\partial \rho(\vec{r}', t_r)}{\partial t_r}\right)$$

$$\vec{\nabla}\rho(\vec{r}', t_r) = \frac{\partial \rho(\vec{r}', t_r)}{\partial t_r}\vec{\nabla} t_r = -\frac{1}{c}\frac{\partial \rho(\vec{r}', t_r)}{\partial t_r}\hat{\mathscr{r}}$$

$$\vec{\nabla}\left(\frac{\partial \rho}{\partial t_r}\right) = -\frac{1}{c}\frac{\partial^2 \rho(\vec{r}', t_r)}{\partial t_r^2}\hat{\mathscr{r}}$$

$$\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$$

$$\vec{\nabla}\cdot\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}$$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int d\tau' \left(\cancel{\frac{1}{c}\frac{\partial \rho(\vec{r}', t_r)}{\partial t_r}\frac{1}{\mathscr{r}^2}} - \rho(\vec{r}', t_r)4\pi\delta^3(\vec{\mathscr{r}}) - \cancel{\frac{1}{c}\frac{1}{\mathscr{r}^2}\frac{\partial \rho(\vec{r}, t_r)}{\partial t_r}} + \frac{1}{c^2}\frac{1}{\mathscr{r}}\frac{\partial^2 \rho(\vec{r}', t_r)}{\partial t_r^2}\right)$$

$$\Rightarrow \nabla^2 V = -\frac{1}{\varepsilon_0}\rho(\vec{r}, \overbrace{t}^{t_r = t \text{ @ } \vec{r}'=\vec{r}}) + \frac{1}{4\pi\varepsilon_0}\overbrace{\mu_0\varepsilon_0}^{\frac{1}{c^2}}\int d\tau' \frac{1}{\mathscr{r}}\frac{\partial^2 \rho(\vec{r}', t_r)}{\partial t_r^2}$$

$$\frac{\partial V}{\partial t} = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{1}{\mathscr{r}}\frac{\partial \rho(\vec{r}', t_r)}{\partial t}$$

$$\underbrace{t_r = t - \frac{1}{c}\mathscr{r} \Rightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial t_r}}$$

For an arbitrary pt. $\vec{r}'$
$$\frac{\partial}{\partial t} = \frac{\partial t_r}{\partial t}\frac{\partial}{\partial t_r} = \frac{\partial}{\partial t_r}$$

$$\hookrightarrow \frac{\partial^2 V}{\partial t^2} = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{1}{\mathscr{r}}\frac{\partial^2 \rho(\vec{r}', t_r)}{\partial t_r^2}$$

$$\Rightarrow \nabla^2 V(\vec{r}, t) - \frac{1}{c^2}\frac{\partial^2 V(\vec{r}, t)}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho(\vec{r}, t) \checkmark$$

With an identical derivation applying to the vector potential $\vec{A}$ & its wave equation.

▨ Moving Point Charge

The location of a point charge q @ time t is given by $\vec{w}(t)$. The charge density is:

$$\rho(\vec{r},t) = q\,\delta^3(\vec{r}-\vec{w}(t))$$

In Lorentz gauge the potentials are given by:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d\tau'\,\frac{\rho(\vec{r}',t_r)}{\mathscr{r}} \qquad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d\tau'\,\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}$$

w/ the retarded time $t_r = t - \mathscr{r}/c$ indicating the time @ which the information now reaching $\vec{r}$ ~~about~~ left the pt. $\vec{r}'$, traveling @ speed c.

First evaluate $V(\vec{r},t)$

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\int d\tau'\,\frac{1}{\mathscr{r}}\,\delta^3(\vec{r}'-\vec{w}(t_r))$$

Change of variables:

$$\vec{u} = \vec{r}' - \vec{w}(t_r) \rightarrow d\vec{u} = d\vec{r}' - \underbrace{\frac{\partial\vec{w}(t_r)}{\partial t_r}}_{\vec{v}(t_r)}\,\underbrace{dt_r}_{-\frac{1}{c}\frac{1}{\cancel{2}\mathscr{r}}\cancel{2}\vec{\mathscr{r}}\cdot(-d\vec{r}')}$$

$$= d\vec{r}' - \vec{v}(t_r)\,\frac{1}{c\mathscr{r}}\,\vec{\mathscr{r}}\cdot d\vec{r}'$$

$$\rightarrow d^3u = d\tau'\left(1 - \frac{\vec{v}(t_r)\cdot\vec{\mathscr{r}}}{c\mathscr{r}}\right)$$

So $V(\vec{r},t)$ is:

Peaks @ $\vec{u}=0 \Rightarrow \vec{r}'=\vec{w}(t_r)$, the pt. on path $\vec{w}(t)$ such that $c(t-t_r) = |\vec{r}-\vec{w}(t_r)|$

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\int d^3u\,\frac{1}{\mathscr{r}}\left(1-\frac{\vec{v}(t_r)\cdot\vec{\mathscr{r}}}{c\mathscr{r}}\right)^{-1}\delta^3(\vec{u})$$

$$= \frac{q}{4\pi\varepsilon_0}\,\frac{1}{|\vec{r}-\vec{w}(t_r)|}\cdot\left(1-\frac{\vec{v}(t_r)\cdot(\vec{r}-\vec{w}(t_r))}{c\,|\vec{r}-\vec{w}(t_r)|}\right)^{-1}$$

$$\Rightarrow \boxed{\begin{array}{l} V(\vec{r},t) = \dfrac{qc}{4\pi\varepsilon_0}\cdot\left(c|\vec{r}-\vec{w}(t_r)| - \vec{v}(t_r)\cdot(\vec{r}-\vec{w}(t_r))\right)^{-1} \\ \text{w/ } c(t-t_r) = |\vec{r}-\vec{w}(t_r)| \end{array}}$$

Now, we can obtain $\vec{A}(\vec{r},t)$ using $\vec{J} = \rho\vec{v}$ for a pt. charge:

$$\vec{J}(\vec{r},t) = q\,\vec{v}(t)\,\delta^3(\vec{r}-\vec{w}(t))$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d\tau' \, q\frac{\vec{v}(t_r)\,\delta^3(\vec{r}\,'-\vec{w}(t_r))}{\mathscr{r}}$$

$$= \frac{1}{c^2}\frac{q}{4\pi\varepsilon_0}\int d\tau' \, \frac{\vec{v}(t_r)}{\mathscr{r}}\,\delta^3(\vec{r}\,'-\vec{w}(t_r))$$

The integral is evaluated as before, and gives:

$$\boxed{\vec{A}(\vec{r},t) = \frac{1}{c^2}\vec{v}(t_r)\,V(\vec{r},t)}$$

As an example, consider a charge moving @ constant velocity $\vec{v}$ w/ position $\vec{w}(0) = 0$ @ $t=0 \Rightarrow \vec{w}(t) = \vec{v}t$.

Ex| $\vec{w}(t) = \vec{v}\,t$

First, determine t_r from $c(t-t_r) = |\vec{r}-\vec{v}\,t_r|$

$$c^2(t-t_r)^2 = (\vec{r}-\vec{v}t_r)\cdot(\vec{r}-\vec{v}t_r)$$

$$c^2t_r^2 - 2c^2 t t_r + c^2t^2 = r^2 + v^2t_r^2 - 2\vec{r}\cdot\vec{v}\,t_r$$

$$\hookrightarrow (c^2-v^2)\,t_r^2 - 2(c^2t-\vec{r}\cdot\vec{v})\,t_r - (r^2-c^2t^2) = 0$$

$$\hookrightarrow t_r = \frac{(c^2t-\vec{r}\cdot\vec{v}) \pm \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)}}{(c^2-v^2)}$$

Which sign, + or − ? $\exists$ 2 pts. on $\vec{w}(t)$ that work; one before t ($t_r < t$) & one after t ($t_r > t$). We want the earlier time, of course. To see which one this corresponds to, look @ the case $\vec{v}=0$:

$$@\,\vec{v}=0?\quad t_r = \frac{c^2t \pm \sqrt{\cancel{c^4t^2} + c^2r^2 - \cancel{c^4t^2}}}{c^2} = t \pm \frac{r}{c}$$

So we want the '−' in this case:

$$t_r = \frac{(c^2t-\vec{r}\cdot\vec{v}) - \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)}}{(c^2-v^2)}$$

$$\rightarrow \overbrace{c^2(t-t_r)}^{c|\vec{r}-\vec{v}t|} - \vec{v}\cdot(\vec{r}-\vec{v}t) = \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)}$$

$$\Rightarrow V(\vec{r},t) = \frac{qc}{4\pi\varepsilon_0}\cdot\left((c^2t-\vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)\right)^{-1/2}$$

$$\vec{A}(\vec{r},t) = \frac{1}{c^2}\vec{v}\,V(\vec{r},t)$$

What are $\vec{E}$ & $\vec{B}$ for the moving point charge? Given the potentials V & $\vec{A}$ we need to evaluate

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

This will require expressions for $\vec{\nabla} t_r$, & the derivatives of functions of t_r. First we derive $\vec{\nabla} t_r$:

$$c(t - t_r) = |\vec{r} - \vec{w}(t_r)| \Rightarrow -c\vec{\nabla} t_r = \vec{\nabla}|\vec{r} - \vec{w}(t_r)|$$

$$\vec{\nabla}|\vec{r}-\vec{w}(t_r)| = \frac{1}{2|\vec{r}-\vec{w}(t_r)|} \vec{\nabla}\left((\vec{r}-\vec{w}(t_r)) \cdot (\vec{r}-\vec{w}(t_r))\right)$$

$$= \frac{1}{\cancel{2}|\;\;|}\Big(\cancel{2}(\vec{r}-\vec{w}(t_r)) \times (\vec{\nabla} \times (\vec{r}-\vec{w}(t_r)))$$
$$+ \cancel{2}((\vec{r}-\vec{w}(t_r)) \cdot \vec{\nabla})(\vec{r}-\vec{w}(t_r))\Big)$$

$$= \frac{1}{|\;\;|}\Big((\vec{r}-\vec{w}(t_r)) \times (\cancelto{0}{\vec{\nabla} \times \vec{r}} - \vec{\nabla} \times \vec{w}(t_r)) + (\vec{r} \cdot \vec{\nabla})(\vec{r} - \vec{w}(t_r))$$
$$- (\vec{w}(t_r) \cdot \vec{\nabla})(\vec{r} - \vec{w}(t_r))\Big)$$

(i) $(\vec{r} \cdot \vec{\nabla})\vec{r} = \vec{r}$

(ii) $(\vec{r} \cdot \vec{\nabla})\vec{w}(t_r) = \dfrac{\partial \vec{w}(t_r)}{\partial t_r}\, \vec{r} \cdot \vec{\nabla} t_r = \vec{v}(t_r)\, \vec{r} \cdot \vec{\nabla} t_r$

(iii) $(\vec{w}(t_r) \cdot \vec{\nabla})\vec{r} = \vec{w}(t_r)$

(iv) $(\vec{w}(t_r) \cdot \vec{\nabla})\vec{w}(t_r) = \dfrac{\partial \vec{w}(t_r)}{\partial t_r}\, \vec{w}(t_r) \cdot \vec{\nabla} t_r = \vec{v}(t_r)\, \vec{w}(t_r) \cdot \vec{\nabla} t_r$

(v) $\vec{\nabla} \times \vec{w}(t_r) = \left(\dfrac{\partial}{\partial y} w_z(t_r) - \dfrac{\partial}{\partial z} w_y(t_r)\right)\hat{x} + \ldots$

$$= \left(\frac{\partial t_r}{\partial y} v_z(t_r) - \frac{\partial t_r}{\partial z} v_y(t_r)\right)\hat{x} + \ldots$$

$$= (\vec{\nabla} t_r) \times \vec{v}(t_r)$$

$$\vec{\nabla}|\vec{r}-\vec{w}(t_r)| = \frac{1}{|\;\;|}\Big(-(\vec{r}-\vec{w}(t_r)) \times ((\vec{\nabla} t_r) \times \vec{v}(t_r))$$
$$+ \vec{r} - \vec{v}(t_r)\, \vec{r} \cdot \vec{\nabla} t_r - \vec{w}(t_r) + \vec{v}(t_r)\, \vec{w}(t_r) \cdot \vec{\nabla} t_r\Big)$$

$$= \frac{1}{|\;\;|}\Big(\vec{r} - \vec{w}(t_r) - \vec{v}(t_r)\,(\vec{r}-\vec{w}(t_r)) \cdot \vec{\nabla} t_r$$
$$\underbrace{- (\vec{r} - \vec{w}(t_r)) \times ((\vec{\nabla} t_r) \times \vec{v}(t_r))}_{\vec{v}(t_r)(\vec{r}-\vec{w}(t_r)) \cdot \vec{\nabla} t_r - (\vec{\nabla} t_r)((\vec{r}-\vec{w}(t_r)) \cdot \vec{v}(t_r))}\Big)$$

Using $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$\overbrace{\vec{\nabla}|\vec{r}-\vec{w}(t_r)|}^{-c\vec{\nabla}t_r} = \frac{1}{|\vec{r}-\vec{w}(t_r)|}\left(\vec{r}-\vec{w}(t_r) - \left((\vec{r}-\vec{w}(t_r))\cdot\vec{v}(t_r)\right)\vec{\nabla}t_r\right)$$

$$\hookrightarrow -c|\vec{r}-\vec{w}(t_r)|\,\vec{\nabla}t_r = \vec{r}-\vec{w}(t_r) - \left((\vec{r}-\vec{w}(t_r))\cdot\vec{v}(t_r)\right)\vec{\nabla}t_r$$

$$\Rightarrow \vec{\nabla}t_r = \frac{-(\vec{r}-\vec{w}(t_r))}{c|\vec{r}-\vec{w}(t_r)| - (\vec{r}-\vec{w}(t_r))\cdot\vec{v}(t_r)}$$

Now begin to calculate $-\vec{\nabla}V$:

$$-\vec{\nabla}V = \frac{qc}{4\pi\varepsilon_0}\left(\quad\right)^{-2}\left(c\vec{\nabla}|\vec{r}-\vec{w}(t_r)| - \vec{\nabla}\left(\vec{v}(t_r)\cdot(\vec{r}-\vec{w}(t_r))\right)\right)$$

Introduce the notation $\vec{R}_r = \vec{r}-\vec{w}(t_r)$. Then

$$\vec{\nabla}t_r = \frac{-\vec{R}_r}{cR_r - \vec{R}_r\cdot\vec{v}(t_r)} \qquad \frac{\partial\vec{R}_r}{\partial t_r} = -\vec{v}(t_r)$$

$$-\vec{\nabla}V = \frac{qc}{4\pi\varepsilon_0}\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-2}\left(\underbrace{c\vec{\nabla}R_r}_{\substack{-c^2\vec{\nabla}t_r \text{ since} \\ c(t-t_r)=R_r}} - \underbrace{\vec{\nabla}(\vec{v}(t_r)\cdot\vec{R}_r)}_{\substack{\vec{v}(t_r)\times(\vec{\nabla}\times\vec{R}_r) + \vec{R}_r\times(\vec{\nabla}\times\vec{v}(t_r)) \\ +(\vec{v}(t_r)\cdot\vec{\nabla})\vec{R}_r + (\vec{R}_r\cdot\vec{\nabla})\vec{v}(t_r)}}\right)$$

(i) $\vec{v}(t_r)\times(\vec{\nabla}\times\vec{R}_r) = \vec{v}(t_r)\times(\cancelto{0}{\vec{\nabla}\times\vec{r}} - \vec{\nabla}\times\vec{w}(t_r))$

$$= -\vec{v}(t_r)\times\left((\vec{\nabla}t_r)\times\vec{v}(t_r)\right)$$

$$= -v(t_r)^2\,\vec{\nabla}t_r + \vec{v}(t_r)(\vec{v}(t_r)\cdot\vec{\nabla}t_r)$$

(ii) $\vec{R}_r\times(\vec{\nabla}\times\vec{v}(t_r)) = \vec{R}_r\times((\vec{\nabla}t_r)\times\vec{a}(t_r))$

$$= (\vec{R}_r\cdot\vec{a}(t_r))\vec{\nabla}t_r - \vec{a}(t_r)(\vec{R}_r\cdot\vec{\nabla}t_r)$$

(iii) $(\vec{v}(t_r)\cdot\vec{\nabla})\vec{R}_r = \vec{v}(t_r) - \vec{v}(t_r)(\vec{v}(t_r)\cdot\vec{\nabla}t_r)$

(iv) $(\vec{R}_r\cdot\vec{\nabla})\vec{v}(t_r) = \vec{a}(t_r)(\vec{R}_r\cdot\vec{\nabla}t_r)$

$$-\vec{\nabla}V = \frac{qc}{4\pi\varepsilon_0}\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-2}\Big(-c^2\vec{\nabla}t_r + v(t_r)^2\vec{\nabla}t_r \cancel{-\vec{v}(t_r)(\vec{v}(t_r)\cdot\vec{\nabla}t_r)}$$
$$-(\vec{R}_r\cdot\vec{a}(t_r))\vec{\nabla}t_r \cancel{+\vec{a}(t_r)(\vec{R}\cdot\vec{\nabla}t_r)}$$
$$-\vec{v}(t_r) \cancel{+\vec{v}(t_r)(\vec{v}(t_r)\cdot\vec{\nabla}t_r)}$$
$$\cancel{-\vec{a}(t_r)(\vec{R}_r\cdot\vec{\nabla}t_r)}\Big)$$

$$-\vec{\nabla} V = \frac{qc}{4\pi\varepsilon_0}\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-2}\left[-\vec{v}(t_r) - \left(c^2 - v(t_r)^2 + \vec{R}_r\cdot\vec{a}(t_r)\right)\vec{\nabla} t_r\right]$$

$$\Rightarrow\ -\vec{\nabla} V = -\frac{qc}{4\pi\varepsilon_0}\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-2}\left[\vec{v}(t_r) + \left(c^2 - v(t_r)^2 + \vec{R}_r\cdot\vec{a}(t_r)\right)\vec{\nabla} t_r\right]$$

And using $\vec{\nabla} t_r = -\vec{R}_r\left(cR_r - \vec{R}_r\cdot\vec{v}(t_r)\right)^{-1}$ we get

$$-\vec{\nabla} V = -\frac{qc}{4\pi\varepsilon_0}\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-3}\Big[\vec{v}(t_r)\left(cR_r - \vec{R}_r\cdot\vec{v}(t_r)\right) - \vec{R}_r\left(c^2 - v(t_r)^2 + \vec{R}_r\cdot\vec{a}(t_r)\right)\Big]$$

Next, we need $-\partial\vec{A}/\partial t$:

$$-\frac{\partial\vec{A}}{\partial t} = -\frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{v}(t_r)\,V(\vec{r},t)\right) = -\frac{1}{c^2}\overbrace{\frac{\partial t_r}{\partial t}}^{\left(1-\frac{\vec{v}(t_r)\cdot\vec{R}_r}{cR_r}\right)^{-1},\ \text{from } c\Delta t = |\vec{r}-\vec{w}(t_r)|}\left(\vec{a}(t_r)\,V(\vec{r},t) + \vec{v}(t_r)\frac{\partial V(\vec{r},t)}{\partial t_r}\right)$$

$$= -\frac{1}{c^2}\frac{cR_r}{(cR_r - \vec{v}(t_r)\cdot\vec{R}_r)}\left(\vec{a}(t_r)\,V(\vec{r},t) + \vec{v}(t_r)\frac{\partial V(\vec{r},t)}{\partial t_r}\right)$$

$$V(\vec{r},t) = \frac{qc}{4\pi\varepsilon_0}\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-1}$$

$$\frac{\partial V}{\partial t_r} = \frac{qc}{4\pi\varepsilon_0}(-1)\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-2}\left(c\frac{\partial R_r}{\partial t_r} - \vec{a}(t_r)\cdot\vec{R}_r + \vec{v}(t_r)\cdot\vec{v}(t_r)\right)$$

$$= -\frac{qc}{4\pi\varepsilon_0}\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-2}\left(-c\frac{\vec{R}_r\cdot\vec{v}(t_r)}{R_r} - \vec{a}(t_r)\cdot\vec{R}_r + v(t_r)^2\right)$$

$$-\frac{\partial\vec{A}}{\partial t} = -\frac{1}{\cancel{c^2}}\frac{\cancel{c}R_r}{(cR_r - \vec{v}(t_r)\cdot\vec{R}_r)}\frac{q\cancel{c}}{4\pi\varepsilon_0}\Big(\vec{a}(t_r)\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-1} - \left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-2}\left(-c\vec{v}\frac{\vec{R}_r\cdot\vec{v}}{R_r} - \vec{v}\,\vec{a}\cdot\vec{R}_r + \vec{v}\,v^2\right)\Big)$$

$$= -\frac{q}{4\pi\varepsilon_0}\left(cR_r - \vec{v}\cdot\vec{R}_r\right)^{-3}R_r\left(\vec{a}\left(cR_r - \vec{v}\cdot\vec{R}_r\right) + c\vec{v}\frac{\vec{R}_r\cdot\vec{v}}{R_r} + \vec{v}\,\vec{a}\cdot\vec{R}_r - \vec{v}\,v^2\right)$$

$$= -\frac{qc}{4\pi\varepsilon_0}\left(cR_r - \vec{v}\cdot\vec{R}_r\right)^{-3}\left[\frac{R_r}{c}\vec{a}\left(cR_r - \vec{v}\cdot\vec{R}_r\right) + \overbrace{\vec{v}\left(\vec{R}_r\cdot\vec{v}\right)}^{\vec{v}(\vec{R}\cdot\vec{v}-cR)+cR\vec{v}} + \frac{R_r}{c}\left(\vec{a}\cdot\vec{R}_r - v^2\right)\vec{v}\right]$$

$$\Rightarrow\ -\frac{\partial\vec{A}}{\partial t} = -\frac{qc}{4\pi\varepsilon_0}\left(cR_r - \vec{v}(t_r)\cdot\vec{R}_r\right)^{-3}\left[\left(\frac{R_r}{c}\vec{a}_r - \vec{v}_r\right)\left(cR_r - \vec{v}_r\cdot\vec{R}_r\right) + \frac{R_r}{c}\left(c^2 - v_r^2 + \vec{a}_r\cdot\vec{R}_r\right)\vec{v}_r\right]$$

$$\vec{E}(\vec{r},t) = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

$$= -\frac{qc}{4\pi\varepsilon_0}(cR_r - \vec{v}_r\cdot\vec{R}_r)^{-3}\Big[(cR_r - \vec{v}_r\cdot\vec{R}_r)\cancel{\vec{v}_r} - \vec{R}_r(c^2 - v_r^2 + \vec{R}_r\cdot\vec{a}_r) + (cR_r - \vec{v}_r\cdot\vec{R}_r)\left(\frac{R_r}{c}\vec{a}_r - \cancel{\vec{v}_r}\right) + \frac{R_r}{c}(c^2 - v_r^2 + \vec{a}_r\cdot\vec{R}_r)\vec{v}_r\Big]$$

$$= -\frac{qc}{4\pi\varepsilon_0}(cR_r - \vec{v}_r\cdot\vec{R}_r)^{-3}\Big[-\left(\vec{R}_r - \frac{R_r}{c}\vec{v}_r\right)(c^2 - v_r^2 + \vec{a}_r\cdot\vec{R}_r) + \frac{R_r}{c}(cR_r - \vec{v}_r\cdot\vec{R}_r)\vec{a}_r\Big]$$

$$= \frac{q}{4\pi\varepsilon_0}(cR_r - \vec{v}_r\cdot\vec{R}_r)^{-3}\Big[(c\vec{R}_r - R\vec{v}_r)(c^2 - v_r^2 + \vec{a}_r\cdot\vec{R}_r) - R_r(cR_r - \vec{v}_r\cdot\vec{R}_r)\vec{a}_r\Big]$$

$$\Rightarrow \boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}(cR_r - \vec{v}_r\cdot\vec{R}_r)^{-3}\Big[(c\vec{R}_r - R_r\vec{v}_r)(c^2 - v_r^2 + \vec{a}_r\cdot\vec{R}_r) - R_r(cR_r - \vec{v}_r\cdot\vec{R}_r)\vec{a}_r\Big]}$$

This can be put in a simpler form by defining the vector $\vec{u}$ as:

$$\vec{u}_r := c\hat{R}_r - \vec{v}_r \quad \leftarrow \vec{v}_r = \vec{v}(t_r)$$

(i) $R_r\vec{u}_r = c\vec{R}_r - R_r\vec{v}_r$

(ii) $\vec{R}_r\cdot\vec{u}_r = cR_r - \vec{R}_r\cdot\vec{v}_r$

(iii) $\vec{R}_r\times(\vec{u}_r\times\vec{a}_r) = \vec{u}_r(\vec{R}_r\cdot\vec{a}_r) - \vec{a}_r(\vec{R}_r\cdot\vec{u}_r)$

$$= \vec{u}_r(\vec{R}_r\cdot\vec{a}_r) - \vec{a}_r(cR_r - \vec{R}_r\cdot\vec{v}_r)$$

$$\hookrightarrow \vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}(\vec{R}_r\cdot\vec{u}_r)^{-3}\Big[R_r\vec{u}_r(c^2 - v_r^2 + \cancel{\vec{a}_r\cdot\vec{R}_r}) + R_r(\vec{R}_r\times\vec{u}_r\times\vec{a}_r) - \cancel{R_r\vec{u}_r(\vec{R}_r\cdot\vec{a}_r)}\Big]$$

$$\Rightarrow \boxed{\begin{aligned} &\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{R_r}{(\vec{R}_r\cdot\vec{u}_r)^3}\Big[(c^2 - v_r^2)\vec{u}_r + \vec{R}_r\times(\vec{u}_r\times\vec{a}_r)\Big] \\ &\text{w/ } \vec{R}_r := \vec{r} - \vec{w}(t_r), \quad \vec{v}_r = \frac{\partial\vec{w}(t_r)}{\partial t_r}, \quad \vec{a}_r = \frac{\partial\vec{v}_r}{\partial t_r} \\ &\text{\& } t_r = t - \frac{1}{c}|\vec{r} - \vec{w}(t_r)| \end{aligned}}$$

For the magnetic field $\vec{B}$ we have:

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A} = \frac{1}{c^2}\vec{\nabla}\times(\vec{v}_r V) = \frac{1}{c^2}\left(V\,\vec{\nabla}\times\vec{v}_r - \vec{v}_r\times(\vec{\nabla}V)\right)$$
$$= \frac{1}{c^2}\left(V\left((\vec{\nabla}t_r)\times\vec{a}_r\right) - \vec{v}_r\times(\vec{\nabla}V)\right)$$

$$V(\vec{r},t) = \frac{qc}{4\pi\varepsilon_0}(\vec{R}_r\cdot\vec{u}_r)^{-1}$$
$$\vec{\nabla}V = \frac{qc}{4\pi\varepsilon_0}(\vec{R}_r\cdot\vec{u}_r)^{-3}\left[\vec{v}_r(\vec{R}_r\cdot\vec{u}_r) - \vec{R}_r(c^2 - v_r^2 + \vec{R}_r\cdot\vec{a}_r)\right]$$
$$\vec{\nabla}t_r = -\vec{R}_r(\vec{R}_r\cdot\vec{u}_r)^{-1}$$

$$\Rightarrow \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{qc}{4\pi\varepsilon_0}\Big((\vec{R}_r\cdot\vec{u}_r)^{-1}\left(-(\vec{R}_r\cdot\vec{u}_r)^{-1}\vec{R}_r\times\vec{a}_r\right) - \ldots$$
$$-(\vec{R}_r\cdot\vec{u}_r)^{-3}\,\vec{v}_r\times\left(\cancel{\vec{v}_r(\vec{R}_r\cdot\vec{u}_r)} - \vec{R}_r(c^2 - v_r^2 + \vec{R}_r\cdot\vec{a}_r)\right)\Big)$$
$$= -\frac{1}{c}\frac{q}{4\pi\varepsilon_0}(\vec{R}_r\cdot\vec{u}_r)^{-3}\left[(\vec{R}_r\cdot\vec{u}_r)\vec{R}_r\times\vec{a}_r + \vec{R}_r\times\vec{v}_r\,(c^2 - v_r^2 + \vec{R}_r\cdot\vec{a}_r)\right]$$
$$= -\frac{1}{c}\vec{R}_r\times\frac{q}{4\pi\varepsilon_0}(\vec{R}_r\cdot\vec{u}_r)^{-3}\Big[\underbrace{(\vec{R}_r\cdot\vec{u}_r)\vec{a}_r}_{\vec{u}_r(\vec{R}_r\cdot\vec{a}_r) - \vec{R}_r\times(\vec{u}_r\times\vec{a}_r)} + (c^2 - v_r^2 + \vec{R}_r\cdot\vec{a}_r)\vec{v}_r\Big]$$
$$= \frac{1}{c}\vec{R}_r\times\frac{q}{4\pi\varepsilon_0}(\vec{R}_r\cdot\vec{u}_r)^{-3}\Big[-\vec{u}_r(\vec{R}_r\cdot\vec{a}_r) + \vec{R}_r\times(\vec{u}_r\times\vec{a}_r)$$
$$+ (c^2 - v_r^2)(-\vec{v}_r) - (\vec{R}_r\cdot\vec{a}_r)\vec{v}_r\Big]$$
$$= \frac{1}{c}\vec{R}_r\times\frac{q}{4\pi\varepsilon_0}(\vec{R}_r\cdot\vec{u}_r)^{-3}\Big[-(\overbrace{\vec{v}_r + \vec{u}_r}^{\text{gives 0 when } \vec{R}\times(\ldots)})(\vec{R}_r\cdot\vec{a}_r) + (c^2 - v_r^2)(\vec{u}_r + \overbrace{c\hat{R}_r}^{\vec{R}_r\times c\hat{R}_r = 0})$$
$$+ \vec{R}_r\times(\vec{u}_r\times\vec{a}_r)\Big]$$

$$\rightarrow \vec{B}(\vec{r},t) = \frac{1}{c}\vec{R}_r\times\Bigg(\underbrace{\frac{q}{4\pi\varepsilon_0}(\vec{R}_r\cdot\vec{u}_r)^{-3}\left[(c^2 - v_r^2)\vec{u}_r + \vec{R}_r\times(\vec{u}_r\times\vec{a}_r)\right]}_{\frac{1}{R_r}\vec{E}(\vec{r},t)}\Bigg)$$

$$\Rightarrow \boxed{\vec{B}(\vec{r},t) = \frac{1}{c}\hat{R}_r\times\vec{E}(\vec{r},t)} \leftarrow \text{w/ } \hat{R}_r = \frac{\vec{R}_r}{R_r}$$

Ex] What are $\vec{E}$ & $\vec{B}$ due to a point charge moving @ constant velocity $\vec{v}$? Assume q is @ the origin @ $t=0$, so $\vec{w}(t) = \vec{v}t$.

$$\vec{R}_r = \vec{r} - \vec{v}t_r \qquad \vec{v}_r = \vec{v} \qquad \vec{a}_r = 0$$

$$t_r = (c^2-v^2)^{-1}\left(c^2t - \vec{r}\cdot\vec{v} - \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)}\right)$$

$$\vec{u}_r = c\hat{R}_r - \vec{v} = c\,\frac{(\vec{r}-\vec{v}t_r)}{|\vec{r}-\vec{v}t_r|} - \vec{v} = \frac{1}{|\vec{r}-\vec{v}t_r|}\left(c\vec{r} - c\cancel{\vec{v}}t_r - \vec{v}\underbrace{|\vec{r}-\vec{v}t_r|}_{c(t-\cancel{t_r})}\right)$$

$$= \frac{1}{|\vec{r}-\vec{v}t_r|}\,c(\vec{r}-\vec{v}t) = c\,\frac{\vec{R}}{R_r}$$

$$\hookrightarrow \vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{R_r}{(\vec{R}_r\cdot\vec{u}_r)^3}(c^2-v^2)\vec{u}_r$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{\cancel{R_r}}{(\vec{R}_r\cdot\vec{u}_r)^3}(c^2-v^2)\,c\,\frac{\vec{R}}{\cancel{R_r}}$$

$$= \frac{q}{4\pi\varepsilon_0}\,c^3\left(1-\frac{v^2}{c^2}\right)\frac{\vec{R}}{(\vec{R}_r\cdot\vec{u}_r)^3}$$

$$\vec{R}_r\cdot\vec{u}_r = cR_r - \vec{R}_r\cdot\vec{v} = c|\vec{r}-\vec{v}t_r| - (\vec{r}-\vec{v}t_r)\cdot\vec{v}$$

$$= c^2(t-t_r) - \vec{v}\cdot(\vec{r}-\vec{v}t_r)$$

via result for t_r, above

$$= \left((c^2t-\vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)\right)^{1/2}$$

$$= \left(\cancel{c^4t^2} + (\vec{r}\cdot\vec{v})^2 - 2\vec{r}\cdot\vec{v}\,c^2t + c^2r^2 - \cancel{c^4t^2} - v^2r^2 + v^2c^2t^2\right)^{1/2}$$

$$= \left(c^2(r^2 - 2\vec{r}\cdot\vec{v}\,t + v^2t^2) + (\vec{r}\cdot\vec{v})^2 - v^2r^2\right)^{1/2}$$

$$= \left(c^2R^2 + (\vec{v}\cdot\vec{R} + v^2t)^2 - v^2r^2\right)^{1/2}$$

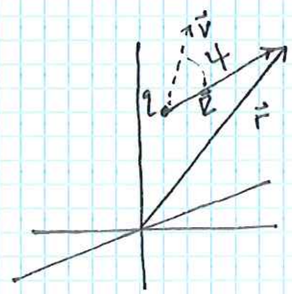
$$= \left(c^2R^2 + (\overbrace{\vec{v}\cdot\vec{R}}^{vR\cos\psi})^2 + v^4t^2 + 2v^2t\,\vec{v}\cdot\vec{R} - v^2r^2\right)^{1/2}$$

$$= \Big(c^2R^2 + v^2R^2 - v^2R^2\sin^2\psi + \cancel{v^4t^2} + 2v^2t\,\vec{v}\cdot\vec{r}$$
$$- \cancel{2}v^4t^2 - v^2r^2\Big)$$

$$= \left(c^2R^2 + v^2R^2 - v^2R^2\sin^2\psi - v^2\overbrace{(\vec{r}-\vec{v}t)\cdot(\vec{r}-\vec{v}t)}^{R^2}\right)^{1/2}$$

$$\rightarrow \vec{R}_r\cdot\vec{u}_r = cR\left(1-\frac{v^2}{c^2}\sin^2\psi\right)^{1/2}$$

w/ ψ the angle b/t $\vec{v}$ & $\vec{R} = \vec{r} - \vec{v}t$



$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\,\cancel{c^3}\left(1-\frac{v^2}{c^2}\right)\frac{\overset{\hat{R}}{\cancel{\vec{R}}}}{\cancel{c^3}\,\underset{R^2}{\cancel{R^3}}}\left(1-\frac{v^2}{c^2}\sin^2\psi\right)^{-3/2}$$

$$\Rightarrow \vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\,\frac{\hat{R}}{R^2}\,\frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}$$

To get $\vec{B}$ we need to evaluate $\hat{R}_r \times \vec{E}$. Start by expressing $\hat{R}_r$ in terms of $\vec{R}$, $\vec{v}$, & R_r:

$$\hat{R}_r = \frac{\vec{R}_r}{R_r} = \frac{1}{R_r}\left(\vec{r}-\vec{v}t_r\right) = \frac{1}{R_r}\left(\vec{r}-\vec{v}t+\vec{v}(t-t_r)\right)$$

$$= \frac{1}{R_r}\left(\vec{R}+\vec{v}\,\frac{1}{c}\,\overbrace{c(t-t_r)}^{R_r}\right)$$

$$\hookrightarrow \hat{R}_r = \frac{R}{R_r}\hat{R} + \frac{\vec{v}}{c}$$

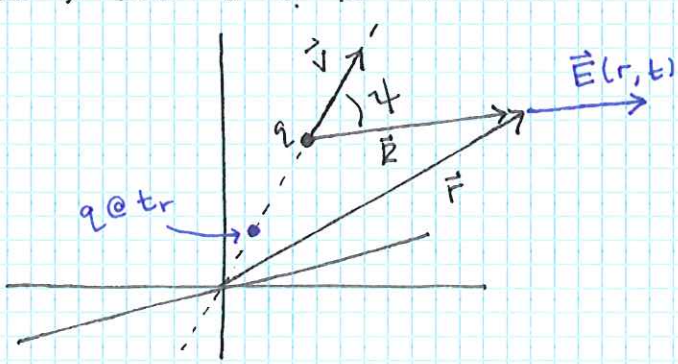
So $\vec{B}(\vec{r},t)$ is

$$\vec{B}(\vec{r},t) = \frac{1}{c}\left(\frac{R}{R_r}\hat{R}+\frac{\vec{v}}{c}\right)\times\overbrace{\vec{E}(\vec{r},t)}^{\sim\hat{R}} = \frac{1}{c^2}\,\vec{v}\times\vec{E}(\vec{r},t)$$

Thus, the Electric & Magnetic fields for a pt. charge q moving @ constant velocity, w/ pos. $\vec{w}(t)=\vec{v}t$, are:

$$\boxed{\begin{aligned}\vec{E}(\vec{r},t) &= \frac{q}{4\pi\varepsilon_0}\,\frac{\hat{R}}{R^2}\,\frac{\left(1-v^2/c^2\right)}{\left(1-\frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}\\ \vec{B}(\vec{r},t) &= \frac{\mu_0 q}{4\pi}\,\frac{(\vec{v}\times\hat{R})}{R^2}\,\frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}\end{aligned}}$$

w/ ψ the $\measuredangle$ b/t $\vec{v}$ & $\vec{R}=\vec{r}-\vec{v}t$



A few points about $\vec{E}$ & $\vec{B}$ for a pt. charge w/ constant $\vec{v}$:

(1) $\vec{E}(\vec{r},t)$ is due to the influence of q @ time t_r, before t, when it was located @ $\vec{w}(t_r)$. Nevertheless, the direction of $\vec{E}(\vec{r},t)$ is <u>not</u> $\hat{R}_r$, but $\hat{R}$: the unit vector pointing from the charges <u>current</u> location (time t) to $\vec{r}$.

(2) The 'inverse distance squared' factor is likewise R^{-2}, and <u>not</u> R_r^{-2}.

(3) However, $\vec{E}(\vec{r},t)$ is not the Coulomb result $(q/4\pi\varepsilon_0)\,\hat{R}/R^2$. There is an additional factor that depends on v & also the direction of $\vec{v}$ relative to $\hat{R}$:

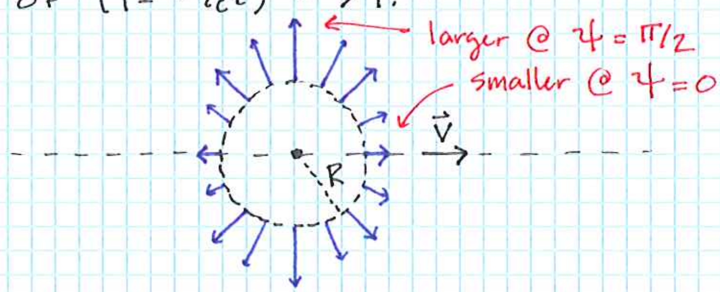
$$f(v,\psi) = \frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}$$

$\frac{\partial}{\partial\psi} f(v,\psi) = 0$ @ $\psi = 0, \pm\pi/2, \pi$

$f(v,0) = f(v,\pi) = 1 - \frac{v^2}{c^2}$

$f(v,\pm\pi/2) = \left(1 - v^2/c^2\right)^{-1/2}$

So @ points <u>along</u> the path of q ($\psi = 0$ or $\psi = \pi$) the field is <u>smaller</u> by a factor of $1 - v^2/c^2 < 1$. When $\hat{R}$ is perp. to $\vec{v}$ ($\psi = \pm\pi/2$) the field is <u>larger</u> by a factor of $(1 - v^2/c^2)^{-1/2} > 1$.



(4) $\vec{B}(\vec{r},t)$ is $\perp$ to $\vec{v}$ & $\vec{E}(\vec{r},t)$ (or $\vec{v}$ & $\hat{R}$). At any instant t we can set up 'instantaneous cylindrical polar coords' w/ $\hat{z} = \hat{v}$ & $\hat{R} = \sin\psi\,\hat{s} + \cos\psi\,\hat{z}$. Then:

$$\vec{v} \times \hat{R} = v\,\hat{z} \times (\sin\psi\,\hat{s} + \cos\psi\,\hat{z}) = v\sin\psi\,\hat{z}\times\hat{s}$$
$$= v\sin\psi\,\hat{\phi}$$

In these coords $\vec{B}(\vec{r},t)$ is:

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{qv}{R^2}\,\frac{(\sin\psi)(1 - v^2/c^2)}{\left(1 - \frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}\,\hat{\phi}$$

So $\vec{B}(\vec{r},t)$ is further suppressed @ $\psi = 0, \pi$, w/ $\vec{B} = 0$ directly in front of or behind q on its path.