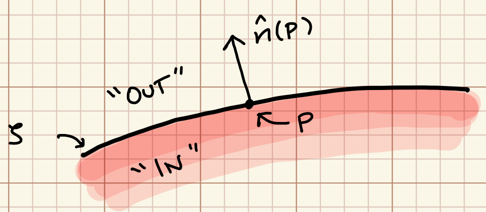
Surface Charge & Discontinuity in $\vec{E}$

- In class we said that the electric field is discontinuous @ a pt. on a surface where there's a surface charge density.

- Let S be some surface, open or closed, with a surface charge density on it. P is a specific point of the surface, $\hat{n}(P)$ is a unit vector perp. to S @ pt. P, and $\sigma(P)$ is the surface charge density @ that pt. Whatever direction $\hat{n}$ is pointing, we'll call the region on that side of S "OUT" & the region on the other side "IN".

- Then the exact statement is



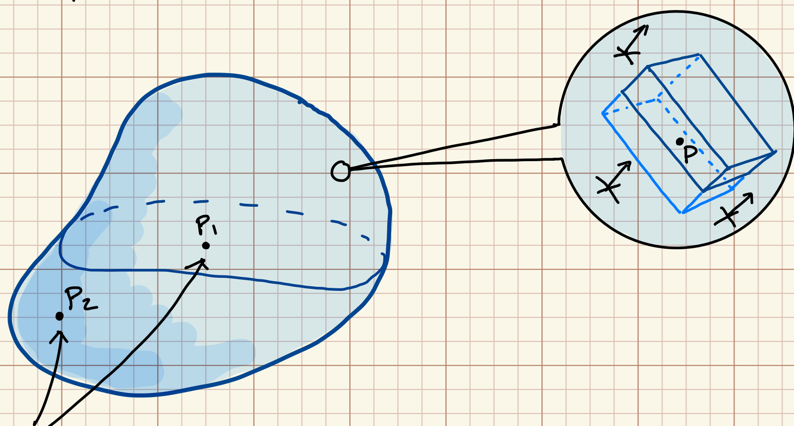
$$\underbrace{\lim_{\vec{r}\to P^{out}} \vec{E}(\vec{r})}_{\text{Behavior of } \vec{E} \text{ as we approach } P \text{ from "OUT" SIDE}} - \underbrace{\lim_{\vec{r}\to P^{in}} \vec{E}(\vec{r})}_{\text{Behavior of } \vec{E} \text{ as we approach } P \text{ from "IN" side}} = \underbrace{\frac{\sigma(P)}{\varepsilon_0}\hat{n}(P)}_{\text{They don't agree if } \sigma(P)\neq 0!}$$

- Usually we have some description of $\vec{E}(\vec{r})$ & $V(\vec{r})$ on either side of S, maybe from Gauss's Law or some other calculation. Then we usually just write:

$$\underbrace{\vec{E}_{out}(P) - \vec{E}_{in}(P) = \frac{\sigma(P)}{\varepsilon_0}\hat{n}(P)}_{\text{Our results for } \vec{E} \text{ on either side of the surface don't agree at the surface.}} \qquad \underbrace{V_{out}(P) = V_{in}(P)}_{\text{However, the potential is always continuous.}}$$

- Now how do we arrive @ this result for the discontinuity in E?

- Start w/ any surface, open or closed, with some charge on its surface described by a charge density σ. It doesn't have to be constant - it might vary from point-to-point on the surface!



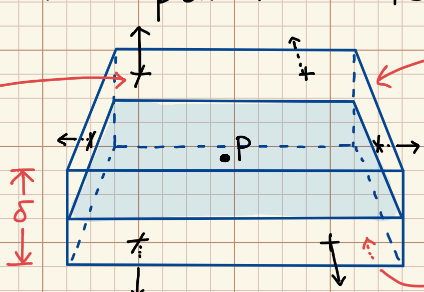
... But if we zoom in on a sufficiently small patch of surface, σ should be more or less constant over that tiny patch. Likewise, if it's a really small patch then $\hat{n}$ is approximately constant!

Some pts have more charge, some have less...

$\sigma(P_2) > \sigma(P_1)$

- We zoom in on a tiny patch of surface around a pt. P. By "tiny" we mean small enough that both σ & $\hat{n}$ are approximately constant over the tiny patch of surface.

- Now look @ a tiny Gaussian surface shaped like a box w/ thickness δ perp. to the surface & faces of area 'A' parallel to the surface.



- By construction (because of how we set it up), the little box encloses a patch of surface w/ area A that has an approximately constant charge density on it equal to $\sigma(P)$. So our Gaussian surface encloses charge $q_{enc} \simeq \sigma(P) \times A$.

- What's the flux of $\vec{E}$ through this box?

$$\Phi_E = \oint_{G.S.} d\vec{a} \cdot \vec{E} = \int_{Top} da\, \hat{n}_{top} \cdot \vec{E} + \int_{Bottom} da\, \hat{n}_{bot} \cdot \vec{E} + \int_{Sides} da\, \hat{n}_{sides} \cdot \vec{E}$$

- Let's look @ each part. For the top, the face is parallel to our little patch of surface so $\hat{n}_{top} = \hat{n}(P)$. Since $\sigma \simeq$ constant, we'd expect $\vec{E}$ is also approximately constant over the face: $\vec{E} \simeq \vec{E}_{out}(\delta/2)$. So:

$$\int_{Top} da\, \hat{n}_{top} \cdot \vec{E} \simeq A\, \hat{n}(P) \cdot \vec{E}_{out}(\delta/2)$$

- The same argument applies to the bottom, except there $\hat{n}_{bot} = -\hat{n}(P)$ (it needs to point from inside our G.S. to outside) and $\vec{E} \simeq \vec{E}_{in}(-\delta/2)$:

$$\int_{Bot} da\, \hat{n}_{bot} \cdot \vec{E} \simeq -A\, \hat{n}(P) \cdot \vec{E}_{in}(-\delta/2)$$

- Finally, the sides involve the parts of $\vec{E}$ parallel to our little patch of surface. However, if I let δ be <u>very</u> small I expect their contribution to the flux should be proportional to δ.

$$\int_{Sides} da\, \hat{n}_{sides} \cdot \vec{E} = \delta \times (\text{stuff})$$

If δ is small, $\hat{n} \cdot \vec{E}$ doesn't vary much from pt. to pt. on the sides & $\Phi_E \propto$ area which is $\propto \delta$!

- Now, do we expect δ to be very small? YES.
We want to know about any discontinuities in $\vec{E}$ @ the surface, which means a finite difference in $\vec{E}$ @ two infinitesimally nearby points. So as we let $\delta \to 0$ we get:

$$\lim_{\delta\to 0} \oint_{a.s.} da\, \hat{n}\cdot\vec{E} = \lim_{\delta\to 0} A\hat{n}(P)\cdot\vec{E}_{out}\left(\frac{\delta}{2}\right) + \lim_{\delta\to 0} A\left(-\hat{n}(P)\right)\cdot\vec{E}_{bot}(-\delta/2)$$

$$+ \underbrace{\lim_{\delta\to 0} \delta \times (\text{stuff})}$$

As $\delta \to 0$, flux through sides is negligible (for $\delta \ll \sqrt{A}$ it is much smaller than top & bottom contributions)

$$\lim_{\delta\to 0} \frac{q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\sigma(P)\times A$$

$\longleftarrow$ All charge on surface, so any δ no matter how small means $q_{enc} = \sigma(P)A$.

GAUSS: $\cancel{A}\,\hat{n}(P)\cdot\vec{E}_{out}(P) - \cancel{A}\,\hat{n}(P)\cdot\vec{E}_{in}(P) = \cancel{A}\,\frac{\sigma(P)}{\varepsilon_0}$

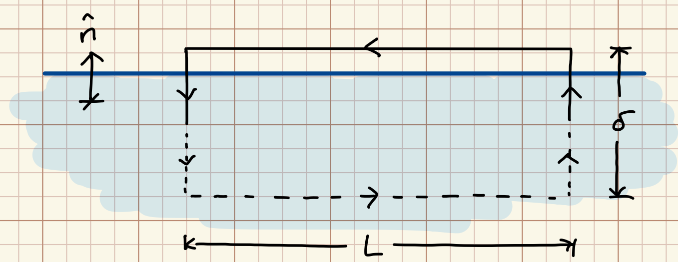
- This tells us that the part of $\vec{E}$ perp. to the surface has a discontinuity @ P. But what about the other components of $\vec{E}$, which are parallel to the surface @ P?

- We know that $\vec{\nabla}\times\vec{E} = 0$ in electrostatics, so STOKES'S THEOREM tells us

$$\int_S da\,\hat{n}\cdot\underbrace{(\vec{\nabla}\times\vec{E})}_{0} = \oint_{\partial S} d\vec{l}\cdot\vec{E} = 0 \quad \text{since } \vec{\nabla}\times\vec{E} = 0$$

FOR ANY CLOSED LOOP.

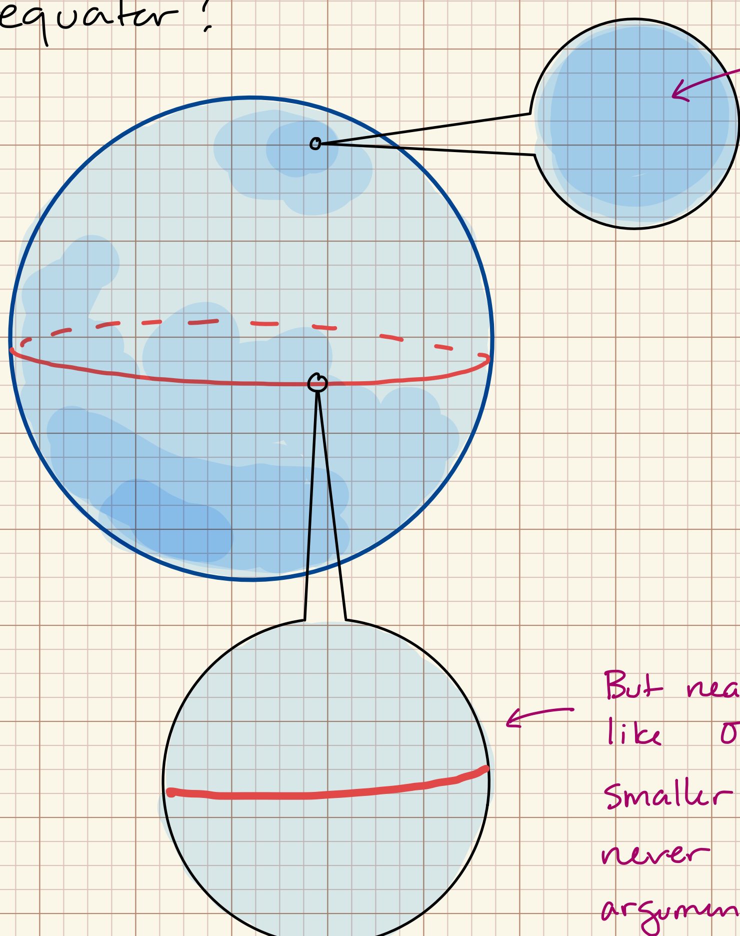
- We can use this to figure out what happens to the other components of $\vec{E}$ @ the surface. Just look @ a closed rectangular loop w/ segments of length L parallel to $\mathcal{S}$ & segments of length δ perp. to $\mathcal{S}$:



- In electrostatics, integrating $d\vec{l}\cdot\vec{E}$ around this closed loop has to give zero, since $\vec{\nabla}\times\vec{E}=0$.
- The sides perp. to $\mathcal{S}$ cancel each other. So for the whole thing to equal zero, the contributions from the two segments parallel to the surface have to cancel each other. As $\delta\to 0$ this only works if the parts of $\vec{E}$ parallel to $\mathcal{S}$ have the same value infinitesimally above & below the surface.
- Therefore, the parts of $\vec{E}$ parallel to $\mathcal{S}$ are continuous. They don't exhibit any funny behavior because of σ.
- Putting this all together – $\hat{n}\cdot\vec{E}$ has a discontinuity @ $\mathcal{S}$ but not the components of $\vec{E}$ parallel to $\mathcal{S}$ – we have:

$$\vec{E}_{out}(P)-\vec{E}_{in}(P)=\frac{\sigma(P)}{\varepsilon_0}\hat{n} \qquad V_{out}(P)=V_{in}(P)$$

- This is true @ any point P on S, even if σ changes as we move around S. So if $\sigma(P_2) > \sigma(P_1)$, the "jump" in the normal component of $\vec{E}$ is larger @ P_2 than @ P_1.
- However, we made some assumptions deriving this result!
- It's always true that if I zoom in far enough on a smooth surface, I find a little patch where $\hat{n}$ is basically constant. Is that true for σ?
- What if I had something like a sphere w/ a smooth surface charge $\sigma(\theta,\phi)$, as well as a line charge density that wraps around the equator?



"Smooth" means $\sigma(\theta,\phi)$ can vary from pt. to pt., but not too abruptly. If P & P' are infinitesimally close then $\sigma(P')$ is only infinitesimally larger or smaller than $\sigma(P)$. We can always zoom in & find a tiny patch around P where $q \sim \sigma(P) \times A$, and our derivation works.

But near a pt on the equator, σ looks like $\sigma(\theta,\phi) + \lambda\,\delta(\theta - \pi/2)$. Zooming in on smaller & smaller areas, we never find $q \sim (\text{const}) \times A$. Our argument breaks down.

- So <u>really</u> what we showed is that the normal component of $\vec{E}$ experiences a finite discontinuity $\sigma(P)/\varepsilon_0$ at any point P where σ is "smooth." Our criteria for smooth is basically that if I look @ a small enough area A around the pt. P I find $q \simeq \sigma(P) \times A$. If the surface also has something funny like a discontinuity in σ @ P, or a line or pt. charge @ P, our argument breaks down.