**SURFACE CHARGE & DISCONTINUITY IN \( \vec{E} \)**

- In class we said that the electric field is discontinuous at a pt. on a surface where there's a surface charge density.

- Let \( S \) be some surface, open or closed, with a surface charge density on it. \( P \) is a specific point of the surface, \( \hat{n}(P) \) is a unit vector perp. to \( S \) @ pt. \( P \), and \( \sigma(P) \) is the surface charge density @ that pt. Whatever direction \( \hat{n} \) is pointing, we'll call the region on that side of \( S \) "OUT" & the region on the other side "IN".

- Then the exact statement is

\[
\lim_{\vec{r} \to \text{OUT}} \vec{E}(\vec{r}) - \lim_{\vec{r} \to \text{IN}} \vec{E}(\vec{r}) = \frac{\sigma(P)}{\varepsilon_0} \hat{n}(P)
\]

Behavior of \( \vec{E} \) Behavior of \( \vec{E} \) They don't agree as we approach as we approach if \( \sigma(P) \neq 0 \)

\( P \) from "OUT" \( P \) from "IN" side

- Usually we have some description of \( \vec{E}(\vec{r}) \) & \( V(\vec{r}) \) on either side of \( S \), maybe from Gauss's Law or some other calculation. Then we usually just write:

\[
\vec{E}_\text{out}(P) - \vec{E}_\text{in}(P) = \frac{\sigma(P)}{\varepsilon_0} \hat{n}(P) \quad V_\text{out}(P) = V_\text{in}(P)
\]

Our results for \( \vec{E} \) on either side of the surface don't agree at the surface.

However, the potential is always continuous.
- Now how do we arrive at this result for the discontinuity in $E$?
- Start w/ any surface, open or closed, with some charge on its surface described by a charge density $\sigma$. It doesn't have to be constant - it might vary from point-to-point on the surface!

... But if we zoom in on a sufficiently small patch of surface, $\sigma$ should be more or less constant over that tiny patch. Likewise, if it's a really small patch then $\hat{n}$ is approximately constant!

Some pts have more charge, some have less...

$\sigma(P_2) > \sigma(P_1)$

- We zoom in on a tiny patch of surface around a pt. $P$. By "tiny" we mean small enough that both $\sigma$ & $\hat{n}$ are approximately constant over the tiny patch of surface.

- Now look at a tiny Gaussian surface shaped like a box w/ thickness $\delta$ perp. to the surface & faces of area `A` parallel to the surface.
- By construction (because of how we set it up), the little box encloses a patch of surface w/ area \( A \) that has an approximately constant charge density on it equal to \( \sigma(P) \). So our Gaussian surface encloses charge \( q_{\text{enc}} = \sigma(P) \times A \).

- **What's the flux of \( \vec{E} \) through this box?**

\[
\Phi_E = \oint_{\text{g.s.}} \vec{d}a \cdot \vec{E} = \int_{\text{Top}} \vec{n}_{\text{top}} \cdot \vec{E} + \int_{\text{Bottom}} \vec{n}_{\text{bot}} \cdot \vec{E} + \int_{\text{Sides}} \vec{n}_{\text{sides}} \cdot \vec{E}
\]

- Let's look at each part. For the top, the face is parallel to our little patch of surface so \( \vec{n}_{\text{top}} = \hat{n}(P) \). Since \( \sigma \approx \text{constant} \), we'd expect \( \vec{E} \) is also approximately constant over the face: \( \vec{E} \approx \vec{E}_{\text{out}}(\delta/2) \). So:

\[
\int_{\text{Top}} \vec{n}_{\text{top}} \cdot \vec{E} = A \hat{n}(P) \cdot \vec{E}_{\text{out}}(\delta/2)
\]

- The same argument applies to the bottom, except there \( \vec{n}_{\text{bot}} = -\hat{n}(P) \) (it needs to point from inside or g.s. to outside) and \( \vec{E} \approx \vec{E}_{\text{in}}(-\delta/2) \):

\[
\int_{\text{Bot}} \vec{n}_{\text{bot}} \cdot \vec{E} \approx -A \hat{n}(P) \cdot \vec{E}_{\text{in}}(-\delta/2)
\]

- Finally, the sides involve the parts of \( \vec{E} \) parallel to our little patch of surface. However, if I let \( \delta \) be very small I expect their contribution to the flux should be proportional to \( \delta \):

\[
\int_{\text{Sides}} \vec{n}_{\text{sides}} \cdot \vec{E} = \delta \times \text{(stuff)} \quad \text{if} \ \delta \ \text{is small,} \ \vec{n} \cdot \vec{E} \ \text{doesn't vary much from pt. to pt. on the sides} \Rightarrow \Phi_E \ \text{x area which is} \ \propto \ \delta^2
\]
- Now, do we expect $\delta$ to be very small? Yes. We want to know about any discontinuities in $\vec{E}$ @ the surface, which means a finite difference in $\vec{E}$ @ two infinitesimally nearby points. So as we let $\delta \to 0$ we get:

$$\lim_{\delta \to 0} \frac{1}{\delta} \int da \hat{n} \cdot \vec{E} = \lim_{\delta \to 0} A \hat{n}(P) \cdot \vec{E}_{\text{out}}(\frac{\delta}{2}) + \lim_{\delta \to 0} A (-\hat{n}(P)) \cdot \vec{E}_{\text{bot}}(-\frac{\delta}{2})$$

$$+ \lim_{\delta \to 0} \delta x \text{(stuff)}$$

As $\delta \to 0$, flux through sides is negligible (for $\delta \ll \sqrt{\epsilon_0} i$), is much smaller than top & bottom contributions)

$$\lim_{\delta \to 0} \frac{q_{\text{enc}}}{\delta} = \frac{1}{\epsilon_0} \sigma(P) x A \leftarrow \text{All charge on surface, so any } \delta \text{ no matter how small means } q_{\text{enc}} = \sigma(P) A.$$ 

**Gauss:**

$$A \hat{n}(P) \cdot \vec{E}_{\text{out}}(P) - A \hat{n}(P) \cdot \vec{E}_{\text{in}}(P) = \frac{A \sigma(P)}{\epsilon_0}$$

- This tells us that the part of $\vec{E}$ perp to the surface has a discontinuity @ P. But what about the other components of $\vec{E}$, which are parallel to the surface @ P?

- We know that $\nabla \times \vec{E} = 0$ in electrostatics, so **STOKES'S THEOREM** tells us

$$\int da \hat{n} \cdot (\nabla \times \vec{E}) = \oint_{\partial S} d\vec{E} \cdot \vec{E} = 0 \text{ since } \nabla \times \vec{E} = 0$$

For Any Closed Loop.
- We can use this to figure out what happens to the other components of $\vec{E}$ at the surface. Just look at a closed rectangular loop w/ segments of length $L$ parallel to $\delta$ & segments of length $\delta$ perp. to $\delta$:

- In electrostatics, integrating $d\vec{l} \cdot \vec{E}$ around this closed loop has to give zero, since $\nabla \times \vec{E} = 0$.

- The sides perp. to $\delta$ cancel each other. So for the whole thing to equal zero, the contributions from the two segments parallel to the surface have to cancel each other. As $\delta \to 0$ this only works if the parts of $\vec{E}$ parallel to $\delta$ have the same value infinitesimally above & below the surface.

- Therefore, the parts of $\vec{E}$ parallel to $\delta$ are continuous. They don't exhibit any funny behavior because of $\delta$.

- Putting this all together: $\hat{n} \cdot \vec{E}$ has a discontinuity @ $\delta$ but not the components of $\vec{E}$ parallel to $\delta$. So we have:

$$\vec{E}_{\text{out}}(P) - \vec{E}_{\text{in}}(P) = \frac{\delta(P)}{\varepsilon_0} \hat{n}$$

$$V_{\text{out}}(P) = V_{\text{in}}(P)$$
- This is true at any point \( P \) on \( S \), even if \( \sigma \) changes as we move around \( S \). So if \( \sigma(P_2) > \sigma(P_1) \), the “jump” in the normal component of \( E \) is larger at \( P_2 \) than at \( P_1 \).

- However, we made some assumptions deriving this result!

- It’s always true that if I zoom in far enough on a smooth surface, I find a little patch where \( \hat{n} \) is basically constant. Is that true for \( \sigma \)?

- What if I had something like a sphere w/ a smooth surface charge \( \sigma(\theta,\phi) \), as well as a line charge density that wraps around the equator?

"Smooth" means \( \sigma(\theta,\phi) \) can vary from pt. to pt., but not too abruptly. If \( P, P' \) are infinitesimally close then \( \sigma(P') \) is only infinitesimally larger or smaller than \( \sigma(P) \). We can always zoom in & find a tiny patch around \( P \) where \( q \sim \sigma(p) \times A \), and our derivation works.

But near a pt on the equator, \( \sigma \) looks like \( \sigma(\theta,\phi) + \lambda \sigma(\theta \rightarrow \frac{\pi}{2}) \). Zooming in on smaller & smaller areas, we never find \( q \sim (\text{const}) \times A \). Our argument breaks down.
So really what we showed is that the normal component of $\vec{E}$ experiences a finite discontinuity $\sigma(P)/\epsilon_0$ at any point $P$ where $\sigma$ is “smooth.” Our criteria for smooth is basically that if I look @ a small enough area $A$ around the pt. $P$ I find $q = \sigma(P) \cdot A$. If the surface also has something funny like a discontinuity in $\sigma @ P$, or a line or pt. charge @ $P$, our argument breaks down.