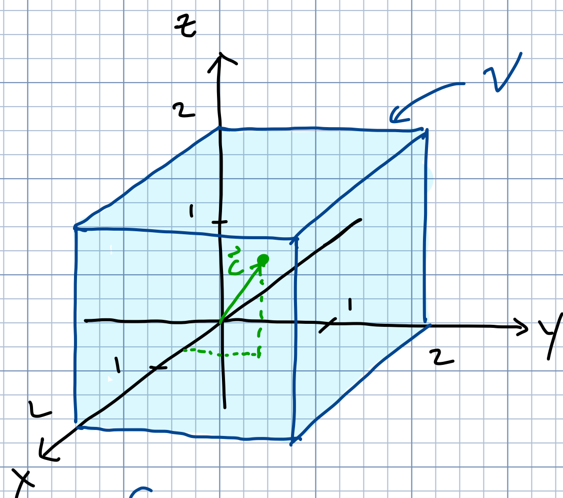
Dirac Delta Examples

(1) Evaluate this integral over a cube w/ $0 \le x, y, z \le 2$.

$$\int_V d\tau \left(\vec{a}\cdot\vec{r} - |\vec{r}|\,\vec{b}\cdot\vec{r}\right)\delta^3(\vec{r}-\vec{c})$$

with $\vec{a} = 4\hat{x} + 7\hat{y} - 3\hat{z}$, $\vec{b} = -5\hat{x} + 2\hat{y} - 11\hat{z}$, and $\vec{c} = \frac{1}{\sqrt{3}}\hat{x} + \frac{1}{\sqrt{3}}\hat{y} + \frac{1}{\sqrt{3}}\hat{z}$.

First, where does $\delta^3(\vec{r}-\vec{c})$ peak? At $\vec{r}=\vec{c}$. Is that inside V? $|\vec{c}| = \left(\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2\right)^{1/2} = 1$, so $\vec{c}$ is a distance 1 from the origin. All components are +, so it is definitely inside the V I described!



Since $\delta^3(\vec{r}-\vec{c})$ has its 'peak' inside V, we just evaluate the rest of the integrand @ $\vec{r}=\vec{c}$.

$$\int_V d\tau \left(\vec{a}\cdot\vec{r} - |\vec{r}|\,\vec{b}\cdot\vec{r}\right)\delta^3(\vec{r}-\vec{c}) = \vec{a}\cdot\vec{c} - |\vec{c}|\,\vec{b}\cdot\vec{c}$$

$$= (4\hat{x}+7\hat{y}-3\hat{z})\cdot\left(\frac{1}{\sqrt{3}}\hat{x} + \frac{1}{\sqrt{3}}\hat{y} + \frac{1}{\sqrt{3}}\hat{z}\right)$$

$$-\left(\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2\right)^{1/2}(-5\hat{x}+2\hat{y}-11\hat{z})\cdot\left(\frac{1}{\sqrt{3}}\hat{x} + \frac{1}{\sqrt{3}}\hat{y} + \frac{1}{\sqrt{3}}\hat{z}\right)$$

$$= \frac{4}{\sqrt{3}} + \frac{7}{\sqrt{3}} - \frac{3}{\sqrt{3}} - 1\times\left(-\frac{5}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{11}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}\times(4+7-3+5-2+11) = \frac{22}{\sqrt{3}}$$

(2) Evaluate

$$\int_V d\tau \left(17x^2 - \sin(302yz) + e^{\frac{z^2}{4}}\right) \delta^3(\vec{r} - 2\hat{x} + 3\hat{y} + 0\hat{z})$$

where V is a sphere of radius 3, centered @ the origin.

Where does $\delta^3(\vec{r} - 2\hat{x} + 3\hat{y} + 0\hat{z})$ peak? At $\vec{r} = 2\hat{x} - 3\hat{y} + 0\hat{z}$. Since

$$|\vec{r}| = \sqrt{2^2 + (-3)^2 + 0^2} = \sqrt{13} \simeq 3.6$$

this is <u>outside</u> V. Therefore the integral is 0.

(3) Evaluate

$$\int_{\substack{\text{All} \\ \text{space}}} d\tau\, \vec{\nabla} \cdot \vec{E} \quad \text{for} \quad \underbrace{\vec{E}(\vec{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}}_{\vec{E} \,@\, \vec{r} \text{ due to } q_1 @ \vec{r}_1{}',\ q_2 @ \vec{r}_2{}', \ldots}, \quad \vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i{}'$$

$$\vec{\nabla} \cdot \vec{E} = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \vec{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}\right) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \times \overbrace{4\pi\, \delta^3(\vec{r} - \vec{r}_i{}')}^{\vec{\nabla} \cdot (\hat{\mathscr{r}}_i / \mathscr{r}_i^2)}$$

$$\int_{\substack{\text{All} \\ \text{Space}}} d\tau \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \delta^3(\vec{r} - \vec{r}_i{}') = \sum_{i=1}^{N} \frac{q_i}{\epsilon_0} \underbrace{\int_{\substack{\text{All} \\ \text{Space}}} d\tau\, \delta^3(\vec{r} - \vec{r}_i{}')}_{}$$

Is $\vec{r} = \vec{r}_i{}'$ in 'All Space'? Yes! So integral $= 1$

$$= \sum_{i=1}^{N} \frac{q_i}{\epsilon_0}$$

$$\Rightarrow \int_{\substack{\text{All} \\ \text{Space}}} d\tau\, \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \sum_{i=1}^{N} q_i$$

Makes sense! $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$, & if we integrate $d\tau\, \rho$ over all space we should get total charge.