The Dirac Delta

1 Point Charges

In E&M we will often talk about the electric field created by charges like the electron. These are true point particles, in the sense that they have properties like mass and charge but no *size* to them.

What does it mean for something to be a point particle? Imagine you start with an object like a baseball: a solid sphere with radius R and mass M. A point particle has no size, so you could imagine taking this sphere and compressing it, retaining all the mass but making it smaller and smaller. As $R \to 0$, you'd say that it has become a point particle.

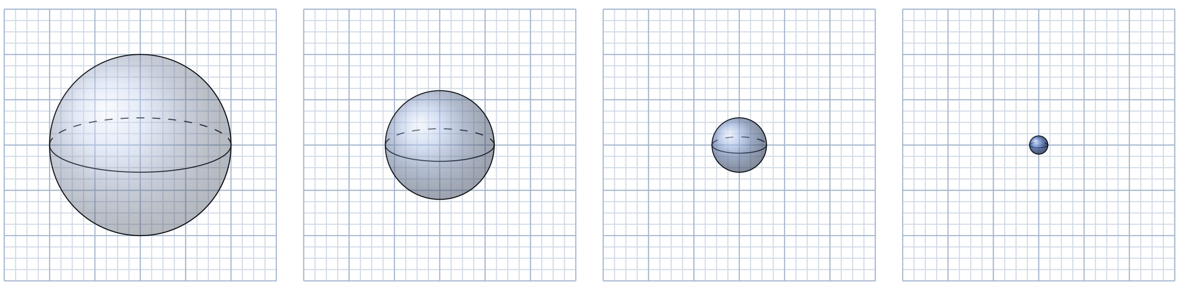
Right away you can see that some of the quantities you might use to describe this object go haywire as $R \to 0$. Assuming the sphere is basically uniform, its density is its mass divided by its volume

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} \ . \tag{1}$$

Let me be a bit more careful and write the density as a function of position. I'll assume that the sphere sits with its center right at the origin, $r = 0$. Then the density is

$$\rho(r) = \begin{cases} \frac{M}{\frac{4}{3}\pi R^3} \ , & r \leq R \\ 0 \ , & r > R \end{cases} \ . \tag{2}$$

In other words, at any point inside the sphere the density is what you'd expect, and at any point outside the sphere, since there is nothing there, the density is zero. Now imagine compressing the sphere, making it smaller and smaller. The same amount of stuff is still there, so the mass doesn't change. But as the radius decreases all that stuff is packed into a much smaller volume and the density increases.



R_1 $\quad$ $R_2 = 0.6 \times R_1$ $\quad$ $R_3 = 0.3 \times R_1$ $\quad$ $R_4 = 0.1 \times R_1$

By the time the radius has reached a tenth of its initial value, the density inside the sphere has grown by a factor of 10^3. As $R \to 0$, 'inside the sphere' just becomes the single point $r = 0$, and the density there diverges: $\lim_{R\to 0} \rho(0) \to \infty$. The concept of density is not well-defined when we are talking about particles that have a finite amount of mass or charge, but no size.

Okay, so why does this matter? You've already seen the formula for the electric field produced by a point charge, and you know that it depends on the amount of charge, not its density. This is true. However, we won't get very far trying to understand the electric fields produced by macroscopic numbers of charges if we have to write them out as the sum of electric fields produced by trillions (or more) of individual point charges. The math would be too cumbersome, even on a computer. As we'll soon see, things will be a lot easier if we imagine large collections of charges as approximately smooth distributions. So if an insulating sphere with a radius of $0.05\,\text{m}$ has roughly 10^{12} extra electrons distributed throughout its volume, we can get a pretty good description of the electric field it produces by treating all that charge as if it were spread evenly throughout the sphere's volume: a uniform charge density $\rho = -3 \times 10^{-4}\,\text{C/m}^3$.

Since we want to work with charge densities, but we also expect to encounter individual charges, it would be nice to have a meaningful way of thinking about the infinite charge density associated with a point charge. The physicist Paul Dirac introduced a means of doing this, which we call the "Dirac delta function". The last part of the name is a misnomer – it's not a function – so I'll just call it the "Dirac delta."

2 The Dirac Delta

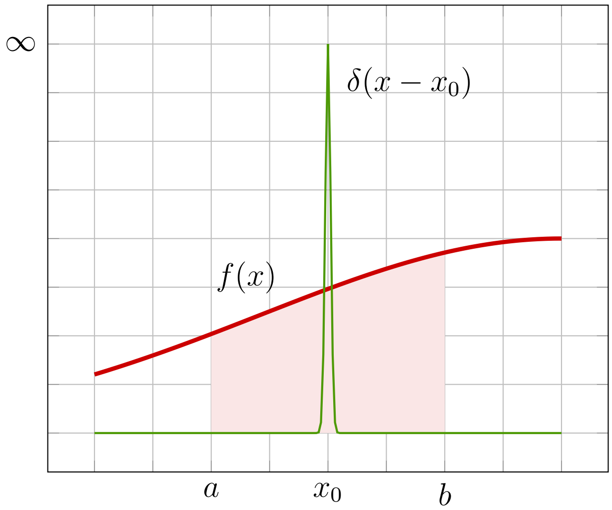
The Dirac delta is a mathematical object called a "distribution." That means that it only makes sense as something that shows up inside an integral alongside an infinitesimally small dx or da or $d\tau$. Nevertheless, let me first try to describe it as one would describe a function, so you can see how it relates to the density of a sphere as we shrink it don to a point particle.

The Dirac delta has an argument, just like a function – we write it as $\delta(x)$. You can think of it as being zero at every point except the one where its argument vanishes (equals zero). Here, for $\delta(x)$, that's at $x = 0$. If instead I considered the Dirac delta $\delta(x - x_0)$, its argument would vanish at $x = x_0$. At the point where its argument vanishes the Dirac delta has an infinitely high peak. This sounds like the density of a point particle from the previous section, right? It's zero almost everywhere, the same way that the density at any point outside of the baseball is zero. But it is infinite at the one point where the point particle is located, much like the density of the baseball at $\rho(0)$ became infinite as we shrunk $R \to 0$. The Dirac delta is the tool we will use when we need to account for the formally infinite density of a point charge in our calculations.

This is where we need to stop thinking about the Dirac delta as a function, and start thinking about it as an object that shows up inside an integral. At the point where its argument vanishes, its height is 'infinite' in the same way that dx is infinitesimally small. If the width dx is basically zero, then the height of the Dirac delta's peak is $1/dx$.

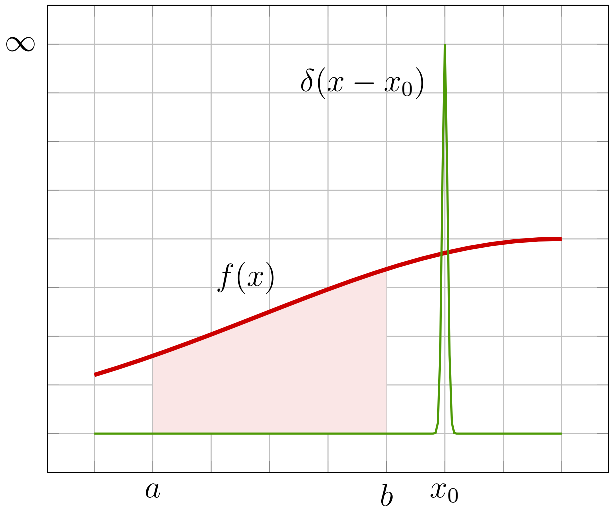
What does this mean? When we evaluate the integral of a function $f(x)$ we move from point-to-point along the x-axis, summing up the areas of thin rectangles with width dx and height $f(x)$. Including a Dirac delta in the integral makes the integrand equal to zero everywhere except for the point $x = x_0$, so almost all the little rectangles now have zero area. But at the point where the Dirac delta peaks, the width of the rectangle is dx and its height is $f(x_0) \times \frac{1}{dx}$, for an area $f(x_0)$.

So here is the rule for evaluating integrals that have a Dirac delta in the integrand: If the peak is located anywhere between the limits of integration, $a < x_0 < b$, then the value of the integral is just the rest of the integrand evaluated at x_0.



$$\int_a^b dx\, f(x)\, \delta(x - x_0) = f(x_0)$$

On the other hand, if the peak occurs outside the limits of integration, then the integral is zero.



$$\int_a^b dx\, f(x)\, \delta(x - x_0) = 0$$

For example, consider the integral of $f(x) = x^4 - 17\,x^3 + 112\,x^2 - 39\,x - 70$ with a Dirac delta that peaks at the point $x = 2$. As long as $x = 2$ falls inside the limits of integration, the result of the integral is just $f(2)$.

$$\begin{aligned}\int_{-14}^{10} dx\, (x^4 - 17\,x^3 + 112\,x^2 - 39\,x - 70)\, \delta(x - 2) &= 2^4 - 17 \times 2^3 + 112 \times 2^2 - 39 \times 2 - 70 \\ &= 180\end{aligned}$$

Notice that we don't evaluate anything at $x = -14$ or $x = 10$, we only asked if the peak of the Dirac delta was located between those two values. If the integral covered the range $x = -2,340$ to $x = 197,635$ we would get the exact same answer. But if the integral ran from $x = 3$ to $x = 10$, the answer would be zero.

3 Dirac Deltas and Changing Variables

The definition of a Dirac delta is

$$\int_a^b dx\, f(x)\, \delta(x - x_0) = \begin{cases} f(x_0)\ , & a < x_0 < b \\ 0 \qquad , & x_0 < a \text{ or } x_0 > b \end{cases} \quad . \tag{3}$$

An important part of this definition is that the variable that shows up in the argument of the Dirac delta is the same as the variable that shows up in the measure (the dx part) of the integral. This insures that the height of $\delta(x - x_0)$ combines with the width dx to form an infinitesimally thin but infinitely tall rectangle with area exactly equal to 1.

However, we will often see integrals where the argument of the Dirac delta is different than our integration variable. Consider an integral of the form

$$\int_{-1}^{1} dx\, f(x)\, \delta(4\, x)\ . \tag{4}$$

The argument of the Dirac delta vanishes at $x = 0$ – the factor of 4 doesn't change that – and this is definitely between the limits of integration. So is this integral just equal to $f(0)$? The answer is "no". The definition (3) tells us how to evaluate integrals with Dirac deltas, and (4) doesn't quite have the same form. We first need to come up with a change of variables that makes (4) look like (3). So define a new variable

$$u = 4\, x \qquad dx = \frac{1}{4}\, du\ . \tag{5}$$

Then (4) is

$$\int_{-1}^{1} dx\, f(x)\, \delta(4\, x) = \frac{1}{4} \int_{-4}^{4} du\, f\left(\frac{u}{4}\right)\, \delta(u)\ . \tag{6}$$

The integral now has the form used in (3). The argument of the Dirac delta vanishes at $u = 0$, which is within the limits of integration, so the integral evaluates as

$$\int_{-1}^{1} dx\, f(x)\, \delta(4\, x) = \frac{1}{4}\, f(0)\ . \tag{7}$$

When the argument of the Dirac delta is related to the integration variable by a constant factor λ, as in $\delta(\lambda\, x)$ or $\delta(\lambda\, (x - x_0))$, the value of the integral gets an overall factor of $1/\lambda$.

Sometimes we will encounter Dirac deltas with arguments that are functions of the integration variable. For instance, we may need to evaluate an integral of the form

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x^2 - 4)\ . \tag{8}$$

This is more complicated than the last example, but as before we handle it by finding a change of variable that makes it look like (3). Actually, it's more accurate to say we use *two* changes of variables, because this Dirac delta has an argument that vanishes at two different points within the region we are integrating over. The two points are located at

$$x^2 - 4 = 0 \;\Rightarrow\; x = 2, -2 \; . \tag{9}$$

Now, remember that an integral can always be split into multiple parts that each cover a portion of the range of integration. For instance, if $a < c < b$, then

$$\int_a^b dx\, f(x) = \int_a^c dx\, f(x) + \int_c^b dx\, f(x) \; . \tag{10}$$

We can do this in the integral (8), splitting into two parts that each contain one of the points where the argument of the Dirac delta vanishes:

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x^2 - 4) = \int_0^{\infty} dx\, f(x)\, \delta(x^2 - 4) + \int_{-\infty}^{0} dx\, f(x)\, \delta(x^2 - 4) \; . \tag{11}$$

Now let's focus on the first term. The argument of the Dirac delta still vanishes at two points, $x = \pm 2$, but only $x = 2$ is between the limits of integration. Let's change the variable to

$$u = x^2 \qquad du = 2\, x\, dx \;\rightarrow\; dx = \frac{1}{2\sqrt{u}}\, du \; . \tag{12}$$

Notice that the last step involved used the square root of $u = x^2$ to write $x = \sqrt{u}$. Remember that a square root can be either positive or negative, $x = \pm\sqrt{u}$, but since the integral is over the region $0 < x < \infty$ where x is positive we clearly want $x = \sqrt{u}$. With the change of variable the first term in (11) is

$$\int_0^{\infty} dx\, f(x)\, \delta(x^2 - 4) = \int_0^{\infty} du\, \frac{1}{2\sqrt{u}}\, f(\sqrt{u})\, \delta(u - 4) \; . \tag{13}$$

The integral now has the same form as our definition (3), but the change of variable has resulted in some new factors in the integrand. The argument of the Dirac delta vanishes at $u = 4$, which is between the limits of integration, so we get

$$\int_0^{\infty} du\, \frac{1}{2\sqrt{u}}\, f(\sqrt{u})\, \delta(u - 4) = \frac{1}{2\sqrt{4}}\, f(\sqrt{4}) = \frac{1}{4}\, f(2) \; . \tag{14}$$

Next, we consider the second integral on the right-hand-side of (11). The same change of variable works here: $u = x^2$. However, since this integral covers the range $-\infty < x < 0$ where x is negative, we want to use the negative root $x = -\sqrt{u}$. Then

$$dx = -\frac{1}{2\sqrt{u}}\, du \; , \tag{15}$$

and the second integral is

$$\int_{-\infty}^{0} dx\, f(x)\, \delta(x^2 - 4) = \int_{\infty}^{0} du \left(-\frac{1}{2\sqrt{u}} \right) f(-\sqrt{u})\, \delta(u - 4) \; . \tag{16}$$

Notice that the lower limit of the integral is now ∞ rather than $-\infty$, since our new variable u is x^2. But we also picked up a minus sign when we wrote dx in terms of du, which we can drop if we flip the limits of integration, so the integral becomes

$$\int_0^\infty du \, \frac{1}{2\sqrt{u}} \, f(-\sqrt{u}) \, \delta(u-4) = \frac{1}{2\sqrt{4}} \, f(-\sqrt{4}) = \frac{1}{4} \, f(-2) \; . \tag{17}$$

Now that we've evaluated both terms on the right-hand-side of (11), we've evaluated the original integral:

$$\int_{-\infty}^\infty dx \, f(x) \, \delta(x^2-4) = \frac{1}{4} \, f(2) + \frac{1}{4} \, f(-2) \; . \tag{18}$$

The two points where the argument of the Dirac delta vanishes both contribute, and there are additional factors of $1/4$ because of the way the integrand changed when we switched over to variables that let us use the definition (3).

When the argument of a Dirac delta is some complicated function of the integration variable, you can always evaluate the integral using the same basic procedure we used above. First, identify all the points where the argument of the Dirac delta vanishes. Second, split the integral into multiple integrals that each contain one of the Dirac delta's peaks. Third, implement a change of variable in each of these sub-integrals that lets you use the definition (3). Finally, add up the results of the individual integrals. As an exercise, try to work out the following result

$$\int_{-\infty}^\infty dx \, f(x) \, \delta(x^2-4x) = \frac{1}{4} \, f(4) + \frac{1}{4} \, f(0) \; . \tag{19}$$

What do you think happens if you change the Dirac delta to $\delta(x^2-4x+4)$?

4 The Dirac Delta in Three Dimensions

So far we've only considered Dirac deltas that depend on a single variable. But the location of a point charge like an electron requires three coordinates. So how do we define a Dirac delta that describes the position of a point particle in three dimensions?

The three-dimensional Dirac delta is just defined as the product of three Dirac deltas

$$\delta^3(x,y,z) = \delta(x) \, \delta(y) \, \delta(z) \; , \tag{20}$$

with a single peak at the point $x=0$, $y=0$, $z=0$ where all three arguments vanish. We often write this as $\delta^3(\vec{r})$, using $\vec{r}$ in place of (x,y,z). If instead we write the Dirac delta $\delta^3(\vec{r}-\vec{r}_0)$, then the peak is located at the point with position vector $\vec{r}=\vec{r}_0$. It works in exactly the same way as before, but now we integrate over some volume $\mathcal{V}$

$$\int_{\mathcal{V}} d\tau \, f(\vec{r}) \, \delta^3(\vec{r}-\vec{r}_0) = \begin{cases} f(\vec{r}_0) \; , & \vec{r}_0 \in \mathcal{V} \\ 0 \; , & \vec{r}_0 \notin \mathcal{V} \end{cases} \; . \tag{21}$$

The volume $\mathcal{V}$ could be a small sphere, or a rectangular box, or literally all of space ($-\infty < x, y, z < \infty$). When evaluating the integral, the only important question is whether the peak at $\vec{r} = \vec{r}_0$ is inside or outside the volume $\mathcal{V}$. If it is inside then the integral is just the rest of the integrand evaluated at $\vec{r} = \vec{r}_0$, and if it is outside then the integral is zero.

As an example, consider the integral of some function $f(x, y, z)$ (it could be anything!) and the Dirac delta $\delta^3(\vec{r})$, over a cube with sides of length 2 centered at the origin. The peak of the Dirac delta is at $\vec{r} = 0$, the origin, which is inside the cube. So

$$\int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz\, f(\vec{r})\, \delta^3(\vec{r}) = f(0, 0, 0)\ . \tag{22}$$

Notice that, just like in the single variable case, the size of the volume $\mathcal{V}$ does not matter. The cube could have sides of length 2 or 1/2 or 10^6. As long as it contains the point $(0, 0, 0)$ then the integral is equal to $f(0, 0, 0)$.

If we repeat this last integral with the Dirac delta $\delta^3(\vec{r} - \frac{1}{2}\,\hat{x} + \frac{1}{4}\,\hat{y} - \frac{3}{8}\,\hat{z})$ then the peak is at $\frac{1}{2}\,\hat{x} - \frac{1}{4}\,\hat{y} + \frac{3}{8}\,\hat{z}$. This point is also inside the cube, so we obtain

$$\int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz\, f(\vec{r})\, \delta^3(\vec{r} - \tfrac{1}{2}\,\hat{x} + \tfrac{1}{4}\,\hat{y} - \tfrac{3}{8}\,\hat{z}) = f(\tfrac{1}{2}, -\tfrac{1}{4}, \tfrac{3}{8})\ . \tag{23}$$

But if we use the Dirac delta $\delta^3(\vec{r} - 2\,\hat{x} - \frac{1}{5}\,\hat{y} + \frac{2}{3}\,\hat{z})$ then the peak is located outside the region covered by the integral and we get

$$\int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz\, f(\vec{r})\, \delta^3(\vec{r} - 2\,\hat{x} - \tfrac{1}{5}\,\hat{y} + \tfrac{2}{3}\,\hat{z}) = 0\ . \tag{24}$$

The function appearing in the integrand does not have to be a scalar. When the integrand is a vector function like $\vec{V}(x, y, z)$ the same rule applies: If the Dirac delta's peak is at a point $\vec{r} = \vec{r}_0$ inside the volume $\mathcal{V}$, then the integral is equal to $\vec{V}(x_0, y_0, z_0)$. Otherwise, it is equal to zero.

Finally, we sometimes encounter integrals where the argument of the three-dimensional Dirac delta is different than the integration variables. For instance, consider the integral

$$\int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz\, f(\vec{r})\, \delta^3(4\,\vec{r})\ . \tag{25}$$

The argument of the Dirac delta vanishes at the origin, which is inside the region we are integrating over. But just like the example we considered in the last section, the argument of the Dirac delta isn't quite the same as our integration variable. It helps to recall the definition (20) here:

$$\delta^3(4\,\vec{r}) = \delta^3(4x\,\hat{x} + 4y\,\hat{y} + 4z\,\hat{z}) = \delta(4x)\,\delta(4y)\,\delta(4z)\ . \tag{26}$$

We already saw what to do when something like $\delta(4x)$ shows up inside an integral over x. Now we just make the same change of variable for all three variables. Each change of variables gives an overall factor of 1/4, so the integral is

$$\int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz\, f(\vec{r})\, \delta^3(4\,\vec{r}) = \frac{1}{4^3}\, f(0, 0, 0)\ . \tag{27}$$