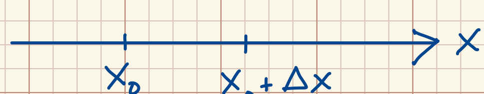
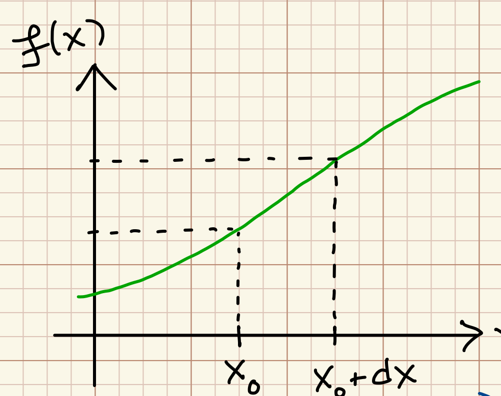
▨ DIFFERENTIALS

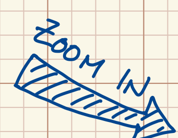
- There's a lot of variation b/t different sections of Calculus, so we're going to review a topic that sometimes gets uneven treatment (or no coverage at all).
- A DIFFERENTIAL is an infinitesimally small change in some quantity.
- It can be something as simple as the 'dx' that separates two infinitesimally nearby points on the x-axis, or it could be the tiny change in some function because of small changes in its arguments.
- (In these notes, words like 'tiny' & 'small' will always mean 'infinitesimal' in the sense you learned about in Calculus!)
- So what <u>is</u> a differential? The 1st differential you learned about was the dx of single variable calculus.
- Consider two nearby points on the x-axis separated by Δx: x_0 and $x_0 + \Delta x$.



If we make Δx very small, the points practically sit on top of each other. When the separation is infinitesimal we replace Δx w/ 'dx'.

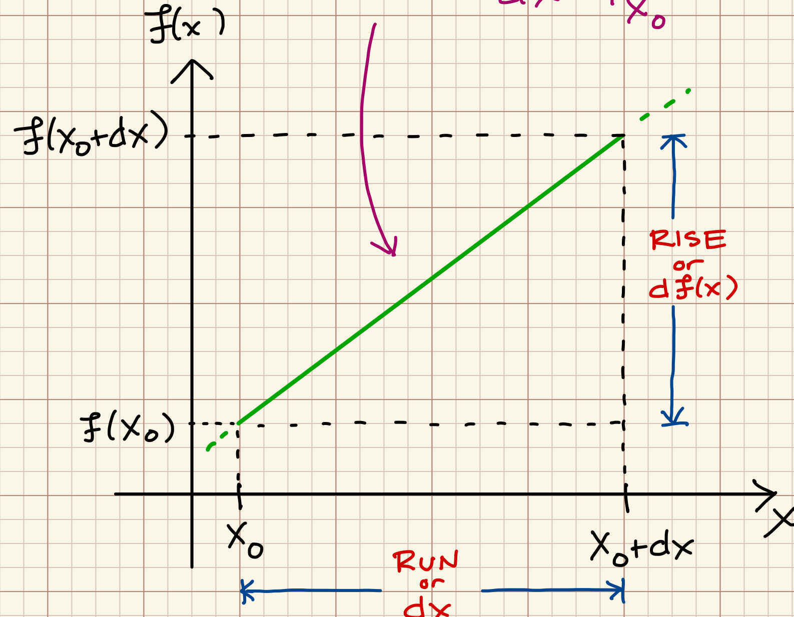
- Now suppose you have some function of the variable x. We'll call it $f(x)$. It could be x^2 or $\sin(4x)$ or anything.
- How does the function change when we move from x_0 over to the infinitesimally nearby point $x_0 + dx$?
- Since dx is 'infinitely small' the function f must not change very much, right?
- Over that tiny distance dx b/t the two points, any well-behaved function is more or less a straight line. As you learned in calc, the slope of that straight line is the DERIVATIVE of $f(x)$ evaluated at x_0.





'EVALUATED AT x_0'

SLOPE $= \left.\dfrac{df(x)}{dx}\right|_{x_0}$



SLOPE = $\dfrac{\text{'RISE'}}{\text{'RUN'}}$

$$\left.\frac{df(x)}{dx}\right|_{x_0} = \frac{f(x_0+dx) - f(x_0)}{dx}$$

$$\Rightarrow f(x_0+dx) = f(x_0) + \left.\frac{df(x)}{dx}\right|_{x_0} dx$$

CHANGED BY THIS AMOUNT

- Since the change in $f(x)$ as we move from x_0 to $x_0 + dx$ is also infinitesimal, we use the same notation we used for dx and write

$$df(x_0) = \left.\frac{df(x)}{dx}\right|_{x_0} dx$$

- Now if we did this for two nearby points x_1 and $x_1 + dx$ we'd likely get a different result, since the derivative of $f(x)$ probably takes different values @ x_0 and x_1:

$$df(x_1) = \left.\frac{df(x)}{dx}\right|_{x_1} dx$$

- The **DIFFERENTIAL** of $f(x)$ tells us how the function changes when we change its argument by an infinitesimal dx:

$$\boxed{df(x) = \frac{df(x)}{dx}\, dx}$$

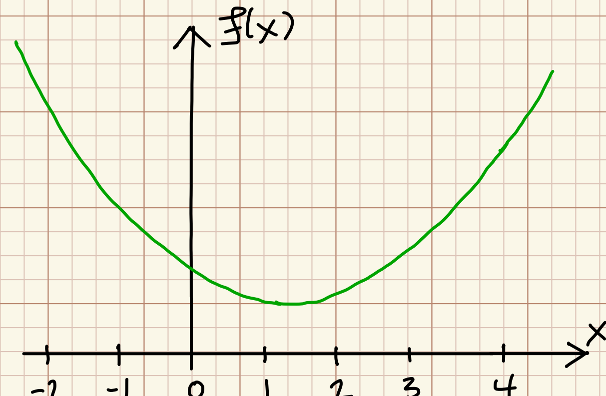
- **EXAMPLE:** $f(x) = x^2 - 3x + 4$

$$\frac{df(x)}{dx} = 2x - 3 \quad \Rightarrow \quad df(x) = (2x-3)\, dx$$

At $x = 0$: $df(0) = -3\, dx$

At $x = 2$: $df(2) = dx$

At $x = 4$: $df(4) = 5\, dx$



Move from 0 to 0+dx & f(x) decreases

Move from 2 to 2+dx, f(x) increases.

- The differential of a function $f(x)$ is just the infinitesimal change in the function when we change its argument by an infinitesimal amant dx.

- What if we changed x by a finite amount? If we move from x_0 to $x_0+\Delta x$ then the change in $f(x)$ is given by the TAYLOR SERIES you learned about in Calculus:

$$f(x_0+\Delta x) = f(x_0) + \left.\frac{df(x)}{dx}\right|_{x_0} \Delta x + \frac{1}{2}\left.\frac{d^2 f(x)}{dx^2}\right|_{x_0} (\Delta x)^2 + \dots$$
$$= f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left.\frac{d^n f(x)}{dx^n}\right|_{x_0} \times (\Delta x)^n$$

- In the limit that $\Delta x \to dx$ becomes infinitesimal we think of $(dx)^2, (dx)^3$, etc as essentially being zero and we're just left with

$$f(x_0+dx) = f(x_0) + df(x_0)$$

- Now let's say we have a function of two variables: $h(x,y)$. How does it change if we move from the point (x_0, y_0) to the nearby point (x_0+dx, y_0+dy)?
- Since the points are infinitesimally close together, the change in h should also be infinitesimal: proportional to dx or dy.

- If we moved from x_0 to x_0+dx but kept y_0 constant, then h would change by

$$h(x_0+dx, y_0) = h(x_0, y_0) + \left.\frac{\partial h(x,y)}{\partial x}\right|_{(x_0,y_0)} dx$$

- Likewise, if we move from y_0 to y_0+dy but keep x_0 constant, the change in h is:

$$h(x_0, y_0+dy) = h(x_0, y_0) + \left.\frac{\partial h(x,y)}{\partial y}\right|_{(x_0,y_0)} dy$$

- And if we change <u>both</u>, we get

$$h(x_0+dx, y_0+dy) = h(x_0, y_0) + \left.\frac{\partial h(x,y)}{\partial x}\right|_{(x_0,y_0)} dx + \left.\frac{\partial h(x,y)}{\partial y}\right|_{(x_0,y_0)} dx$$

- So the differential of $h(x,y)$ is

$$dh(x,y) = \frac{\partial h(x,y)}{\partial x} dx + \frac{\partial h(x,y)}{\partial y} dy$$

- EXAMPLE: $h(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ w/ a,b <u>constant</u>

$$\frac{\partial h(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2}{a^2}\right) + \underbrace{\cancel{\frac{\partial}{\partial x}\left(\frac{y^2}{b^2}\right)}}_{0} - \underbrace{\cancel{\frac{\partial}{\partial x}(1)}}_{0} = \frac{2x}{a^2}$$

$$\frac{\partial h(x,y)}{\partial y} = \frac{2y}{b^2}$$

$$\rightarrow dh(x,y) = \frac{2x}{a^2} dx + \frac{2y}{b^2} dy$$

- Sometimes we'll simplify our expressions a bit & just write

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$$

This means the same thing as before, we're just replacing `h(x,y)' w/ `h' so the expressions aren't so cumbersome.

- Based on the jump from one variable to two you can probably guess how we write the differential of a function that depends on three or more variables:

$$dg(x,y,z) = \frac{\partial g(x,y,z)}{\partial x} dx + \frac{\partial g(x,y,z)}{\partial y} dy + \frac{\partial g(x,y,z)}{\partial z} dz$$

- In physics we often work with VECTOR FUNCTIONS. For example, you might have a force $\vec{F}(x,y)$ with magnitude and/or direction that changes from point to point in the x-y plane:

$$\vec{F}(x,y) = \underbrace{F_x(x,y)}_{\text{The x-component is some function of } x \,\&\, y} \hat{x} + \underbrace{F_y(x,y)}_{\text{The y-component is some other function of } x \,\&\, y.} \hat{y}$$

- What is the differential of something like this?

- It's just like any other function of x & y! If we move from (x_0, y_0) to (x_0+dx, y_0), only changing x, then the change in $\vec{F}$ is

$$\vec{F}(x_0+dx, y_0) = \vec{F}(x_0, y_0) + \left.\frac{\partial \vec{F}(x,y)}{\partial x}\right|_{(x_0,y_0)} dx$$

$\frac{d}{dx}(A+B) = \frac{dA}{dx} + \frac{dB}{dx}$

$$\frac{\partial \vec{F}(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(F_x(x,y)\,\hat{x} + F_y(x,y)\,\hat{y}\right)$$

$$= \underbrace{\frac{\partial}{\partial x}\left(F_x(x,y)\,\hat{x}\right)} + \underbrace{\frac{\partial}{\partial x}\left(F_y(x,y)\,\hat{y}\right)}$$

PRODUCT RULE

$$= \frac{\partial F_x(x,y)}{\partial x}\hat{x} + F_x(x,y)\cancelto{0}{\frac{\partial \hat{x}}{\partial x}} + \frac{\partial F_y(x,y)}{\partial x}\hat{y} + F_y(x,y)\cancelto{0}{\frac{\partial \hat{y}}{\partial x}}$$

$$= \frac{\partial F_x(x,y)}{\partial x}\hat{x} + \frac{\partial F_y(x,y)}{\partial x}\hat{y}$$

← $\hat{x}$ & $\hat{y}$ are constant vectors

$$\vec{F}(x_0+dx, y_0) = \vec{F}(x_0, y_0) + \left(\left.\frac{\partial F_x(x,y)}{\partial x}\right|_{(x_0,y_0)}\hat{x} + \left.\frac{\partial F_y(x,y)}{\partial x}\right|_{(x_0,y_0)}\hat{y}\right)dx$$

- You'd get something similar (involving dy & $\partial/\partial y$) if you move from (x_0, y_0) to (x_0, y_0+dy).
- So the differential of $\vec{F}(x,y)$ is

$$d\vec{F}(x,y) = \left(\frac{\partial F_x(x,y)}{\partial x}\hat{x} + \frac{\partial F_y(x,y)}{\partial x}\hat{y}\right)dx + \left(\frac{\partial F_x(x,y)}{\partial y}\hat{x} + \frac{\partial F_y(x,y)}{\partial y}\hat{y}\right)dy$$

$$= \frac{\partial \vec{F}}{\partial x}dx + \frac{\partial \vec{F}}{\partial y}dy$$

← Like any other function of x & y!

- You could also write this as

$$d\vec{F}(x,y) = \left(\frac{\partial F_x(x,y)}{\partial x} dx + \frac{\partial F_x(x,y)}{\partial y} dy \right) \hat{x} + \left(\frac{\partial F_y(x,y)}{\partial x} dx + \frac{\partial F_y(x,y)}{\partial y} dy \right) \hat{y}$$

$$= dF_x(x,y)\,\hat{x} + dF_y(x,y)\,\hat{y}$$

- Our first expression for $d\vec{F}(x,y)$ looked complicated, but it's the same $dh = \partial h/\partial x \, dx + \partial h/\partial y \, dy$ as before w/ h replaced by $\vec{F}$.

- Likewise, our second expression for $d\vec{F}$ (which is identical to the first, just organized differently) makes sense when we think of $\vec{F}$ changing b/c the two functions F_x & F_y (its components) change.

- EXAMPLE: $\vec{F}(x,y) = \underbrace{(3x + 2y^2 - 4)}_{F_x(x,y)}\hat{x} + \underbrace{(y^3 - 4x^2)}_{F_y(x,y)}\hat{y}$

$$\frac{\partial}{\partial x}(F_x) = 3 \qquad \frac{\partial}{\partial x}(F_y) = -8x$$

$$\frac{\partial}{\partial y}(F_x) = 4y \qquad \frac{\partial}{\partial y}(F_y) = 3y^2$$

$$\hookrightarrow d\vec{F}(x,y) = (3\hat{x} - 8x\hat{y})\,dx + (4y\,\hat{x} + 3y^2\hat{y})\,dy$$

or

$$= (3dx + 4y\,dy)\,\hat{x} + (-8x\,dx + 3y^2dy)\,\hat{y}$$

- Remember that the differential of a vector function is *also* a vector. In this last example we had

$$\vec{F}(x,y) = (3x+2y^2-4)\,\hat{x} + (y^3-4x^2)\,\hat{y}$$

$$d\vec{F}(x,y) = \underbrace{(3x\,dx+4y\,dy)}_{\text{The x-comp. of } d\vec{F} \text{ is } dF_x(x,y).}\,\hat{x} + \underbrace{(-8x\,dx+3y^2dy)}_{\text{The y-comp. of } d\vec{F} \text{ is } dF_y(x,y).}\,\hat{y}$$

- So if you are @ the point $(x_0,y_0)=(1,-2)$ & move to an infinitesimally nearby point $(1+dx,-2+dy)$ the vector $\vec{F}$ changes by

$$d\vec{F}(1,-2) = (3dx-8dy)\,\hat{x} + (-8dx+12dy)\,\hat{y}$$

- Whether we're working with scalar functions like $f(x)$ or $h(x,y)$ (a function that takes 1 or more arguments & returns a number), or vector functions like $\vec{F}(x,y)$, the differential means the same thing. It tells us about the small change in that quantity when its arguments change by an infinitesimal amount.

- Since we get differentials by taking derivatives, we can think of 'd' as something we do to a function that follows the same rules as derivatives.

(1) $d(f+g) = df + dg$

$d(\vec{F}+\vec{G}) = d\vec{F} + d\vec{G}$ } LINEAR

(2) $d(fg) = df\, g + f\, dg$

$d(f\vec{G}) = df\, \vec{G} + f\, d\vec{G}$ } PRODUCT/LEIBNIZ RULE

(3) $d(f(g)) = \frac{df}{dg}\, dg$ } CHAIN RULE

- EXAMPLE: $f(x) = x^2$ & $g(x) = \cos x$

$$d\, f(g(x)) = \frac{d(g(x)^2)}{dg(x)}\, dg(x) = 2g(x)\, dg(x)$$

$$= 2\cos x \cdot (-\sin x)\, dx$$

$$= -2\cos x \sin x\, dx$$

$$f(g(x)) = \cos^2 x \rightarrow d(\cos^2 x) = 2\cos x\, d(\cos x)$$

$$= -2\cos x \sin x\, dx \checkmark$$

- Let's put a few of the things we've seen together in an example.

- EXAMPLE: Instead of using Cartesian coords x & y to describe the plane we can use POLAR COORDINATES ρ & ϕ. They are related to x & y by

$$x(\rho,\phi) = \rho\cos\phi \qquad 0 \le \rho \le \infty$$

$$y(\rho,\phi) = \rho\sin\phi \qquad 0 \le \phi < 2\pi$$

Find the differential of the POSITION VECTOR $\vec{r} = x\hat{x} + y\hat{y}$ in polar coordinates.

$$\vec{r} = x\,\hat{x} + y\,\hat{y} = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y}$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial \rho}\,d\rho + \frac{\partial \vec{r}}{\partial \phi}\,d\phi$$

$$\frac{\partial \vec{r}}{\partial \rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\frac{\partial \vec{r}}{\partial \phi} = -\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y}$$

$$\hookrightarrow d\vec{r}(\rho,\phi) = (\cos\phi\,\hat{x} + \sin\phi\,\hat{y})\,d\rho + (-\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y})\,d\phi$$

$$= (\cos\phi\,d\rho - \rho\sin\phi\,d\phi)\,\hat{x}$$

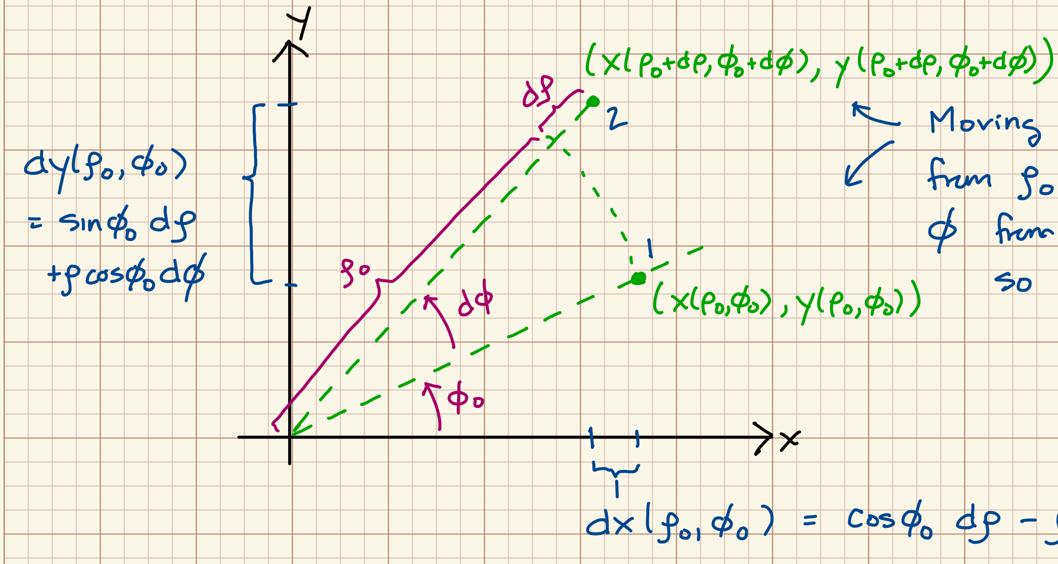
$$+ (\sin\phi\,d\rho + \rho\cos\phi\,d\phi)\,\hat{y}$$

- As part of that example we worked out $dx(\rho,\phi)$ and $dy(\rho,\phi)$:

$$dx = \cos\phi\,d\rho - \rho\sin\phi\,d\phi$$

$$dy = \sin\phi\,d\rho + \rho\cos\phi\,d\phi$$

Do these expressions make sense?



Moving from 1 to 2 increases ρ from ρ_0 to $\rho_0 + d\rho$, then increases ϕ from ϕ_0 to $\phi_0 + d\phi$. 1st quadrant so $\cos\phi_0$ & $\sin\phi_0$ both positive.

- One last example!
- **EXAMPLE:** The distance b/t two infinitesimally separated pts (x,y) & $(x+dx, y+dy)$ is

$$ds = \sqrt{dx^2 + dy^2}$$

by the Pythagorean Theorem. If both points sit on the curve $y(x) = x^2+4$, what is ds?

$$y(x) = x^2+4 \rightarrow dy(x) = 2x\,dx$$

If (x,y) is on the curve $y(x) = x^2+4$, & you move over dx in the x-direction, then y must change by $dy(x) = 2x\,dx$ to stay on the curve!

$$ds = \sqrt{dx^2+dy^2} = \sqrt{dx^2 + (2x\,dx)^2}$$

$$ds = dx\sqrt{1+4x^2}$$

What if both points are on the parameterized curve $x(t) = t^2 \cos t$ & $y(t) = 2 - t \sin t$?

$$dx(t) = 2t \cos t \, dt - t^2 \sin t \, dt = (2t\cos t - t^2 \sin t)\,dt$$

$$dy(t) = -dt \sin t - t \cos t \, dt = -(\sin t + t \cos t)\,dt$$

$\hookrightarrow$ $$ds = \left((2t\cos t - t^2\sin t)^2 + (\sin t + t\cos t)^2\right)^{1/2} dt$$