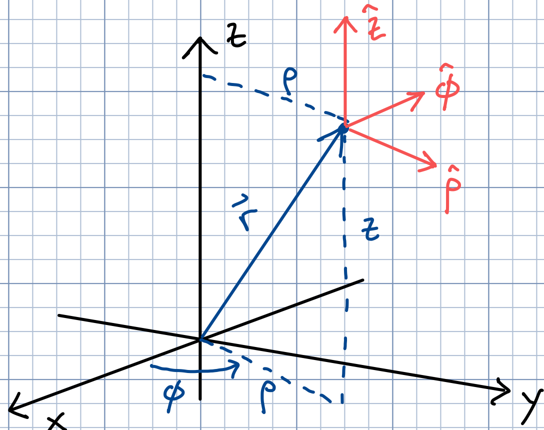
▨ CYLINDRICAL POLAR COORDS: $\vec{r}, \vec{v}, \vec{a}$

- Let's say we want to describe an object as it moves around in 3 dimensions, and for whatever reason (more on this later) we want to use Cylindrical Polar Coordinates (CPC). Let's go through every step.
- First, how are CPC related to Cartesian co-ordinates?



$x = \rho \cos\phi \qquad y = \rho \sin\phi \qquad z = \text{same}$

$\rho = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}(y/x)$

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$
$$= \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\,\hat{z}$$

- Let's work out the scale factors h_ρ, h_ϕ, h_z and unit vectors $\hat{\rho}, \hat{\phi}, \hat{z}$.

Always look for simplifications like $\cos^2 + \sin^2 = 1$!

$$\frac{\partial \vec{r}}{\partial \rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} + 0\,\hat{z} \qquad \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \sqrt{(\cos\phi)^2 + (\sin\phi)^2} = 1$$

$$\Rightarrow \quad h_\rho = 1 \qquad \hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} + 0\,\hat{z}$$

$$\frac{\partial \vec{r}}{\partial \phi} = -\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y} + 0\,\hat{z} \qquad \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi} = \rho$$

$$\Rightarrow \quad h_\phi = \rho \qquad \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} + 0\,\hat{z}$$

$$\frac{\partial \vec{r}}{\partial z} = \hat{z} \quad \Rightarrow \quad h_z = 1 \qquad \hat{z} \text{ is same as Cartesian}$$

- Notice that $\hat{\rho}$ & $\hat{\phi}$ both depend on ϕ when we write them in terms of $\hat{x}$ & $\hat{y}$.

- Next, let's express the position $\vec{r}$ entirely in CPC.

$$\vec{r} = r_\rho \hat{\rho} + r_\phi \hat{\phi} + r_z \hat{z}$$

$$r_\rho = \vec{r} \cdot \hat{\rho} = (\rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\,\hat{z}) \cdot (\cos\phi\,\hat{x} + \sin\phi\,\hat{y} + 0\,\hat{z})$$
$$= \rho(\cos\phi)^2 + \rho(\sin\phi)^2 + \cancel{z \times 0}$$
$$= \rho$$

$$r_\phi = \vec{r} \cdot \hat{\phi} = (\rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\,\hat{z}) \cdot (-\sin\phi\,\hat{x} + \cos\phi\,\hat{y} + 0\,\hat{z})$$
$$= -\rho\cos\phi\sin\phi + \rho\sin\phi\cos\phi + \cancel{z \times 0}$$
$$= 0$$

$$r_z = \vec{r} \cdot \hat{z} = (\rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\,\hat{z}) \cdot \hat{z}$$
$$= z$$

No ϕ component! All the info about ϕ is contained in $\hat{\rho}(\phi) = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$.

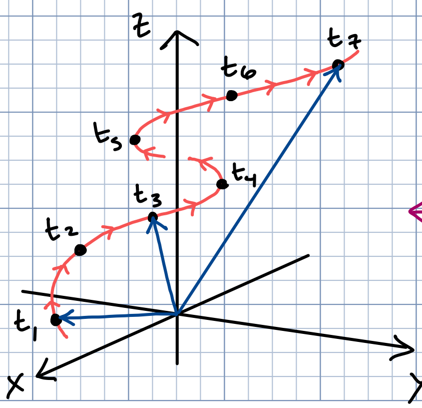
$$\Rightarrow \boxed{\vec{r} = \rho\hat{\rho} + 0\,\hat{\phi} + z\,\hat{z}}$$

- Okay, suppose we're describing an object moving around. At different times it may be getting closer to or further from the z-axis, curling around the axis, or moving up or down parallel to the axis. In other words, ρ, ϕ, & z could all be changing. So we'll say they are functions of t.

$\hat{\rho}$ changes if ϕ changes!

$$\vec{r} = \rho(t)\,\hat{\rho}(\phi(t)) + z(t)\,\hat{z}$$

$\hat{z}$ is constant



The object's position is changing. At different times it may have different positions labeled by different values of ρ, ϕ, & z.

- Now how do we write its velocity? I can think of 3 different ways to work this out. Let's look @ each one.

(1) Start w/ $\vec{r}$ in CPC & take its derivative with respect to time.

$$\vec{v}(t) = \frac{d}{dt}\left(\rho(t)\,\hat{\rho}(\phi(t)) + z(t)\,\hat{z}\right)$$

$$= \underbrace{\frac{d}{dt}\left(\rho(t)\,\hat{\rho}(\phi(t))\right)}_{\text{PRODUCT RULE}} + \underbrace{\frac{d}{dt}\left(z(t)\,\hat{z}\right)}_{\text{PRODUCT RULE}}$$

$$= \frac{d\rho(t)}{dt}\,\hat{\rho}(\phi(t)) + \rho(t)\underbrace{\frac{d\hat{\rho}(\phi(t))}{dt}}_{\text{CHAIN RULE}} + \frac{dz(t)}{dt}\,\hat{z} + z(t)\underbrace{\frac{d\hat{z}}{dt}}_{0}$$

$$\frac{d\phi(t)}{dt}\underbrace{\frac{d\hat{\rho}(\phi)}{d\phi}}_{\frac{d}{d\phi}(\cos\phi\,\hat{x}+\sin\phi\,\hat{y})} = \frac{d\phi(t)}{dt}\;\underbrace{\hat{\phi}(\phi(t))}_{-\sin\phi\,\hat{x}+\cos\phi\,\hat{y}}$$

$$\vec{v}(t) = \frac{d\rho(t)}{dt}\,\hat{\rho}(\phi(t)) + \rho(t)\frac{d\phi(t)}{dt}\,\hat{\phi}(\phi(t)) + \frac{dz(t)}{dt}\,\hat{z}$$

To save space we usually use a dot above a quantity to indicate its time derivative, & we don't explicitly write arguments like (t) or $(\phi(t))$. So

$$\vec{v} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$$

No $\hat{\phi}$ term in $\vec{r}$, but we get one in $\vec{v}$ because $d\hat{\rho}/dt = \dot{\phi}\,\hat{\phi}$!

This is nice and compact, but it's up to us to remember that ρ, ϕ, & z are functions of t, and $\hat{\rho}$ & $\hat{\phi}$ are functions of ϕ!

(2) Start w/ $\vec{r}$ in Cartesian coords, take its time derivative, then use def'ns of CPC & dot products to work out components.

$$\vec{v} = \frac{d}{dt}\left(x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}\right) = \dot{x}\,\hat{x} + \dot{y}\,\hat{y} + \dot{z}\,\hat{z}$$

$$x = \rho\cos\phi \rightarrow \dot{x} = \frac{dx}{dt} = \frac{d}{dt}(\rho\cos\phi) = \frac{d\rho}{dt}\cos\phi + \rho\underbrace{\frac{d\cos\phi}{dt}}_{\frac{d\phi}{dt}\frac{d(\cos\phi)}{d\phi}}$$

CHAIN RULE!

$$= \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi$$

$$y = \rho\sin\phi \rightarrow \dot{y} = \frac{dy}{dt} = \frac{d}{dt}(\rho\sin\phi) = \frac{d\rho}{dt}\sin\phi + \rho\frac{d\sin\phi}{dt}$$

$$= \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi$$

$$\rightarrow \vec{v} = (\dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi)\,\hat{x} + (\dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi)\,\hat{y} + \dot{z}\,\hat{z}$$

$$v_\rho = \vec{v}\cdot\hat{\rho} = \vec{v}\cdot(\cos\phi\,\hat{x} + \sin\phi\,\hat{y} + 0\,\hat{z})$$

$$= (\dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi)\cos\phi + (\dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi)\sin\phi$$

$$= \dot{\rho}\,(\underbrace{\cos^2\phi + \sin^2\phi}_{1})\ \underbrace{-\rho\dot{\phi}\sin\phi\cos\phi + \rho\dot{\phi}\cos\phi\sin\phi}_{0}$$

$$\hookrightarrow v_\rho = \dot{\rho}$$

$$v_\phi = \vec{v}\cdot\hat{\phi} = \vec{v}\cdot(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y} + 0\,\hat{z})$$

$$= (\dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi)(-\sin\phi) + (\dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi)\cos\phi$$

$$= \underbrace{-\dot{\rho}\cos\phi\sin\phi + \dot{\rho}\sin\phi\cos\phi}_{0} + \rho\dot{\phi}\,(\underbrace{\sin^2\phi + \cos^2\phi}_{1})$$

$$\hookrightarrow v_\phi = \rho\dot{\phi}$$

$$\Rightarrow \vec{v} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$$

← Same result we just found, but we computed it differently.

(3) Velocity is the infinitesimal displacement $d\vec{r}$ divided by the infinitesimal dt over which it happened. For any OCS

$$d\vec{r} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3$$

So for CPC ($h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$)

$$d\vec{r} = d\rho\, \hat{\rho} + \rho\, d\phi\, \hat{\phi} + dz\, \hat{z}$$

$$\hookrightarrow \frac{d\vec{r}}{dt} = \frac{d\rho}{dt}\hat{\rho} + \rho \frac{d\phi}{dt}\hat{\phi} + \frac{dz}{dt}\hat{z} = \vec{v}$$

- Of course all 3 calculations give the same $\vec{v}$. The last one is much simpler, right?
- Unfortunately, to work out the object's acceleration we have to follow approach 1 or 2 – there's no shortcut like method 3.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{\rho}(t)\, \hat{\rho}(\phi(t)) + \rho(t)\dot{\phi}(t)\,\hat{\phi}(\phi(t)) + \dot{z}(t)\,\hat{z} \right)$$

(i) $$\frac{d}{dt}\left(\dot{\rho}\,\hat{\rho}(\phi)\right) = \ddot{\rho}\,\hat{\rho}(\phi) + \dot{\rho}\frac{d}{dt}\hat{\rho}(\phi)$$
$$= \ddot{\rho}\,\hat{\rho}(\phi) + \dot{\rho}\frac{d\phi}{dt}\frac{d\hat{\rho}(\phi)}{d\phi}$$
$$= \ddot{\rho}\,\hat{\rho} + \dot{\rho}\dot{\phi}\,\hat{\phi}$$

$$\frac{d}{d\phi}(\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$$
$$= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$= \hat{\phi}$$

(ii) $$\frac{d}{dt}\left(\rho\dot{\phi}\,\hat{\phi}(\phi)\right) = \dot{\rho}\dot{\phi}\,\hat{\phi}(\phi) + \rho\ddot{\phi}\,\hat{\phi}(\phi) + \rho\dot{\phi}\frac{d}{dt}\hat{\phi}(\phi)$$
$$= \dot{\rho}\dot{\phi}\,\hat{\phi} + \rho\ddot{\phi}\,\hat{\phi} + \rho\dot{\phi}\frac{d\phi}{dt}\frac{d\hat{\phi}(\phi)}{d\phi}$$
$$= \dot{\rho}\dot{\phi}\,\hat{\phi} + \rho\ddot{\phi}\,\hat{\phi} + \rho\dot{\phi}^2(-\hat{\rho})$$

$$\frac{d}{d\phi}(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$
$$= -\cos\phi\,\hat{x} - \sin\phi\,\hat{y}$$
$$= -\hat{\rho}$$

$$\Rightarrow \vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

- You could also start w/ $\vec{a} = \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y} + \ddot{z}\,\hat{z}$ in Cartesian coords, use $x = \rho\cos\phi$ & $y = \rho\sin\phi$ to replace $\ddot{x}$ & $\ddot{y}$ w/ derivatives of ρ & ϕ, & then use dot-products of $\vec{a}$ w/ $\hat{\rho}$ & $\hat{\phi}$ to work out a_ρ & a_ϕ.

- Before we do an example, let's collect our results for position, velocity, & acceleration.

Again, notice I'm writing $\dot{\rho}$ instead of $\frac{d\rho(t)}{dt}$, and $\hat{\phi}$ instead of $\hat{\phi}(\phi(t))$, etc.

$$\vec{r} = \rho\,\hat{\rho} + z\,\hat{z}$$

$$\vec{v} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$$

Speed: $|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2}$

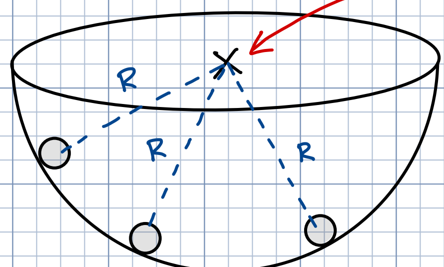
$$\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\rho} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\,\hat{\phi} + \ddot{z}\,\hat{z}$$

- How would we use these results?

- Well, why do we use any OCS besides Cartesian coordinates?

- We use an OCS when it makes describing the system we're studying easier.

- For example, suppose you have a bowl shaped like a hemisphere of radius R & you let a marble roll around inside it.



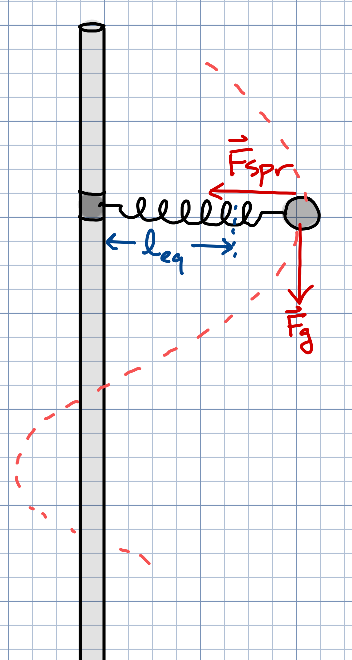
Put the origin here & use SPC. Then $r = R$ and only θ & ϕ change as the marble rolls around. In Cartesian coords, all 3 (x, y, z) would be changing.

- In that case it's the geometry of the system that's telling you to use SPC.

- Sometimes, the right OCS makes it easier to describe forces. For example, suppose you have a mass M connected to a vertical rod by a spring w/ spring constant k. The point where the spring connects to the rod can move up or down, the spring can stretch or be compressed as the mass moves out or in, and the mass can also circle around the rod.

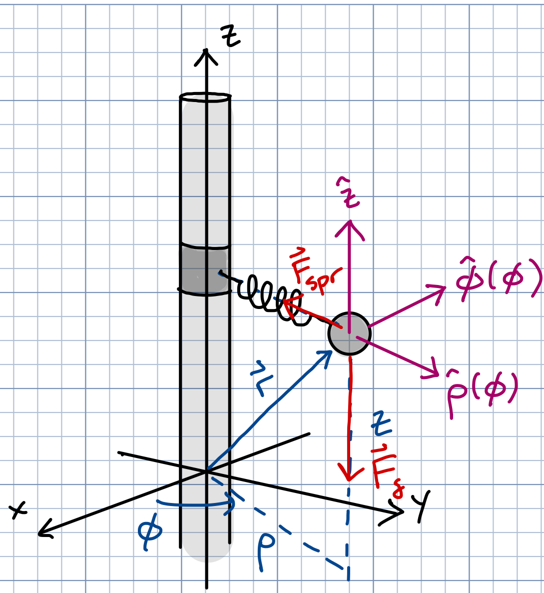
- Sure, this is contrived, but can we write Newton's 2nd Law for the mass?

- There are two forces acting on M: $\vec{F}_g$ pulls down & $\vec{F}_{spr}$ pulls in towards or pushes out from the rod depending on whether the spring is compressed or stretched. We'll use l_{eq} for the equilibrium length of the spring.



The mass will move in the x, y, & z directions. But writing $\vec{F}_{spr}$ would be easier if we had one dir. that always pointed twds / away from the rod.

⇒ Use Cylindrical Polar Coordinates



$$\vec{F}_g = -Mg\,\hat{z}$$

$$\vec{F}_{spr} = \underset{\uparrow}{-}\,k \times \underbrace{(\rho - \ell_{eq})}\,\hat{\rho}$$

Restoring Force...   ... proportional to how stretched out / compressed from equilibrium length

$$\rightarrow \vec{F}_{net} = -Mg\,\hat{z} - k(\rho - \ell_{eq})\,\hat{\rho}$$

- Newton's 2nd law says $\vec{F}_{net} = M\vec{a}$, and we already worked out $\vec{a}$ in CPC:

$$M \times \left((\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\rho} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\,\hat{\phi} + \ddot{z}\,\hat{z} \right) = -Mg\,\hat{z} - k(\rho - \ell_{eq})\,\hat{\rho}$$

ρ-comp: $\quad M \times (\ddot{\rho} - \rho\dot{\phi}^2) = -k(\rho - \ell_{eq})$

ϕ-comp: $\quad M \times (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}) = 0$

z-comp: $\quad M\ddot{z} = -Mg$

- There are lots of solutions to these equations — lots of ways the mass can move — and you'll learn how to solve them in Phys 314. Let's look @ just one of the many possibilities.

- The z-comp. tells us M is pulled down & its vertical motion is freefall ($a_z = -g$):

$$\cancel{M}\ddot{z} = -\cancel{M}g \rightarrow \frac{d^2 z(t)}{dt^2} = -g$$

Starts @ $z = z(t_0)$ w/ $V_z = V_z(t_0)$ @ time t_0.

$$\Rightarrow z(t) = z(t_0) + V_z(t_0) \times (t - t_0) - \frac{1}{2} g \times (t - t_0)^2$$

- The ϕ-component is interesting!

$$M \times \underbrace{(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})}_{= \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\phi})} = 0$$

No force in $\hat{\phi}$ directi

Check: $\frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\phi}) = \frac{1}{\rho}(2\rho\dot{\rho}\dot{\phi} + \rho^2\ddot{\phi})$
$= 2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}$ ✓

$$\Rightarrow \frac{1}{\rho}\frac{d}{dt}(M\rho^2\dot{\phi}) = 0$$

This only works if $M\rho^2\dot{\phi}$ = a constant!

$$\Rightarrow M\rho^2\dot{\phi} = \text{Constant}$$

- This is just the statement that one component of M's Angular Momentum $\vec{L} = \vec{r} \times \vec{p}$ is conserved b/c there's no force in the $\hat{\phi}$ direction.

- We'll call the constant L_z (it's the z-comp. of the angular momentum), so

$$M\rho^2\dot{\phi} = L_z$$

However M is moving, ρ & $\dot{\phi}$ satisfy this eqn!

- Finally, the ρ-component is

$$\ddot{\rho} - \rho\dot{\phi}^2 = -\frac{k}{M}(\rho - l_{eq})$$

Use $\dot{\phi} = \frac{L_z}{M\rho^2}$

$$\ddot{\rho} - \rho\left(\frac{L_z}{M\rho^2}\right)^2 = -\frac{k}{M}(\rho - l_{eq})$$

$$\Rightarrow \ddot{\rho} - \left(\frac{L_z}{M}\right)^2\frac{1}{\rho^3} = -\frac{k}{M}(\rho - l_{eq})$$

- There are lots of sol'ns of this equation depending on how M starts off @ $t = t_0$.

- One solution is ρ = constant, so $\dot{\rho}=0$ & $\ddot{\rho}=0$. For that to work we'd need just the right values of ρ – how far its stretched out – and L_z – how much angular momentum it has from spinning ($\dot{\phi}$) around the rod.

$$\cancelto{0}{\ddot{\rho}} - \rho\dot{\phi}^2 = -\frac{k}{M}(\rho - l_{eq})$$

$$-\left(\frac{L_z}{M}\right)^2 \frac{1}{\rho^3} = -\frac{k}{M}(\rho - l_{eq})$$

- For example, suppose $k = 10^3$ N/m, $l_{eq} = 0.1$ m, and $M = 20$ grams $= 2\times10^{-2}$ kg.

$$-\left(\frac{L_z}{.02\,kg}\right)^2 \frac{1}{\rho^3} = -\frac{10^3\,N/m}{0.02\,kg}\times(\rho - 0.1m)$$

- Suppose you stretch it out so $\rho = 15cm = 0.15m$ & spin it around the rod.

$$\cancel{-}\left(\frac{L_z}{0.02\,kg}\right)^2 \frac{1}{(0.15m)^3} = \cancel{-}\frac{10^3\,N/m}{0.02\,kg}\times(0.05m)$$

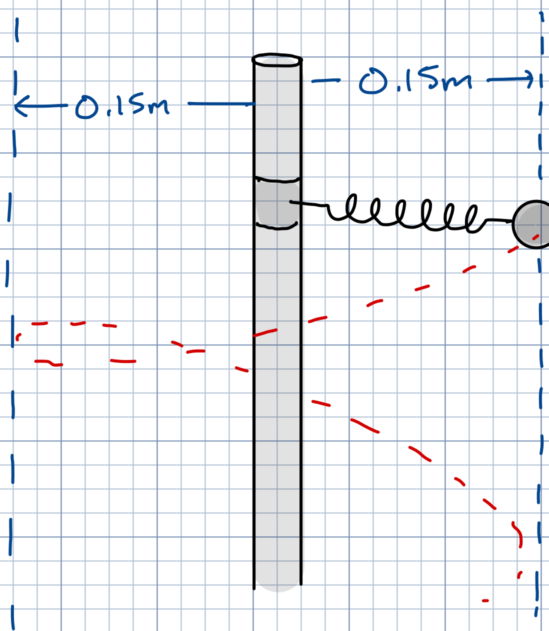
$$\rightarrow L_z = 0.02kg \times \left(\frac{10^3\,N/m}{0.02\,kg}\times 0.05m \times (0.15m)^3\right)^{1/2}$$

$$= 0.0581\ \frac{kg.m^2}{s}$$

$$L_z = 0.0581\ \frac{kg.m^2}{s} = M\rho^2\dot{\phi} = 0.02kg \times (0.15m)^2\,\dot{\phi}$$

$$\Rightarrow \dot{\phi} = 129.1\ rad/s$$

- So $\rho = 0.15m$ & $\dot{\phi} = 129.1$ rad/s – both <u>constant</u> – is one solution of these equations.



For $k = 10^3\,N/m$, $l_{eq} = 0.1m$, and $M = 0.02\,kg$, if you pull the mass out to $\rho = 0.15m$ & spin it around the rod @ $\dot{\phi} = 129.1\,rad/s$ it will move in a circle (ρ won't change) as it drops.

- This makes sense! For constant $\rho = 0.15m$ & $\dot{\phi} = 129.1\,rad/s$ the $\hat{\phi}$ comp. of $\vec{V}$ is $V_\phi = \rho\dot{\phi} = 19.37\,m/s$. Maintaining that constant speed requires a centripeta acceleration $V_\phi^2/\rho = 2.5\times10^3\,m/s^2$. That's exactly what we get from the spring!

$$\vec{F}_{spr} = -10^3\frac{N}{m}\times(0.15m - 0.1m)\,\hat{\rho} = -50N\,\hat{\rho}$$

$$\rightarrow \frac{\vec{F}_{spr}}{M} = -\frac{50N}{0.02\,kg}\,\hat{\rho} = -2500\,\frac{m}{s^2}\,\hat{\rho}$$

- Looking back @ what we did, & keeping in mind that $\rho \geq 0$ in CPC, can you see why this kind of solution doesn't exist for $\rho < l_{eq}$?
- Again, this example is a little contrived. And there are many more complicated sol'ns to the equations we worked out. The point is just to show you that even that first step – writing out $\vec{F}_{net} = M\vec{a}$ – may be easier w/ an OCS!