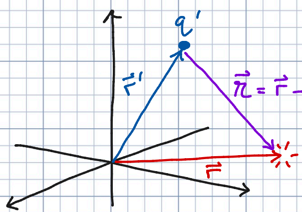
Coulomb Integrals

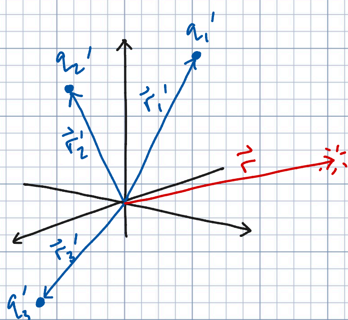
- The electric field @ the point with position vector $\vec{r}$ due to a (point) charge q' located @ the point w/ position vector $\vec{r}\,'$ is:



$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$\vec{E}(\vec{r}) = \frac{q'}{4\pi\varepsilon_0}\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{q'}{4\pi\varepsilon_0}\,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

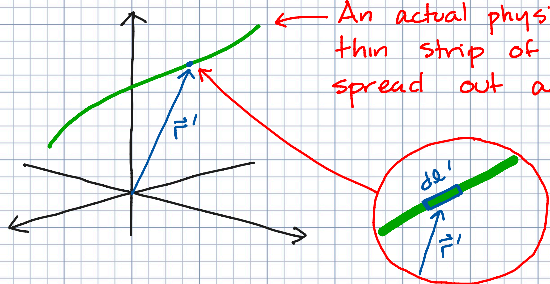
- If there are N charges q_i', $i = 1, \ldots, N$, located @ points with position vectors $\vec{r}_i\,'$, then the <u>total</u> electric field @ the point w/ position vector $\vec{r}$ is the <u>sum</u> of the field produced by each charge:



$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} \frac{q_i'}{4\pi\varepsilon_0}\,\frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^{\,2}} = \sum_{i=1}^{N} \frac{q_i'}{4\pi\varepsilon_0}\,\frac{\vec{r}-\vec{r}_i\,'}{|\vec{r}-\vec{r}_i\,'|^3}$$

- In principle, this is all of electrostatics. But finding $\vec{E}$ for realistic numbers of point charges is intractable so we usually replace collections of many charges w/ <u>distributions</u> of charge. The way the charge is spread out along a curve, surface, or volume is characterized by a <u>charge density</u>.

- For example, spread a total charge Q along a line or curve. The <u>line charge density</u> $\lambda(\vec{r}\,')$ tells you how much charge is found along a tiny length $d\ell'$ located @ $\vec{r}\,'$:



An actual physical object like a wire or thin strip of plastic w/ total charge Q spread out along it.

The bit of length $d\ell'$ located @ $\vec{r}\,'$ has charge $dq(\vec{r}\,') = \lambda(\vec{r}\,')\, d\ell'$.

- Notice that $\lambda(\vec{r}\,')$ depends on $\vec{r}\,'$ — that is, different bits along the length of the curve may have more or less charge than the other bits. If we add up all the charges for each bit along the path we get the total charge:

$dq(\vec{r}\,') = d\ell'\,\lambda(\vec{r}\,')$

$$Q = \int_P dq(\vec{r}\,') = \int_P d\ell'\,\lambda(\vec{r}\,')$$

Visit every point along P & add up the charges.

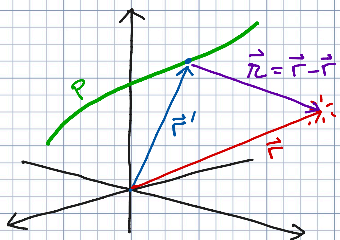
- If the charge is spread <u>uniformly</u> along P then λ is a constant & each bit has the same amount of charge on it:

$$Q = \int_P d\ell'\,\lambda \overset{\text{If constant}}{=} \lambda \overbrace{\int_P d\ell'}^{\text{length!}} = \lambda L \Rightarrow \lambda = \frac{Q}{L}$$

Special case of a <u>uniform</u> line charge density.

- Be Careful! Don't pull charge densities out of integrals if they are not constant!

- To find the electric field @ a point w/ position vector $\vec{r}$ produced by charge spread out along a path P, we just add up the contributions from all the little charges $dq(\vec{r}\,')$ like we would w/ point charges.



Visit every pt. along P

Add $\vec{E}$ produced @ $\vec{r}$ by charge $dq(\vec{r}\,')$ to total

$$\vec{E}(\vec{r}) = \int_P \overbrace{\frac{1}{4\pi\varepsilon_0}\, dq(\vec{r}\,')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}$$

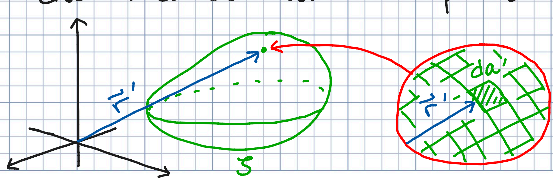
$$= \frac{1}{4\pi\varepsilon_0} \int_P dl'\, \lambda(\vec{r}\,')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_P dl'\, \lambda(\vec{r}\,')\, \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

- This is <u>exactly</u> the same thing we did when we had N point charges. Except now, $N \to \infty$ and a point charge q_i' located @ $\vec{r}_i\,'$ becomes an infinitesimal bit of charge $dq(\vec{r}\,')$ labeled by its position $\vec{r}\,'$ along the path. The sum over an ∞ number of infinitesimal quantities becomes an integral.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_P dl'\, \lambda(\vec{r}\,')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

- What if we spread a charge Q over some surface $\mathcal{S}$? The <u>surface charge density</u> $\sigma(\vec{r}\,')$ tells us how much charge is present on an infinitesimal area da' located at the point $\vec{r}\,'$ on the surface:



The patch of area da' @ point $\vec{r}\,'$ on the surface has charge $dq(\vec{r}\,') = da'\,\sigma(\vec{r}\,')$

- As with the line charge, if we add up the charges on each bit of the surface we get the total charge Q.

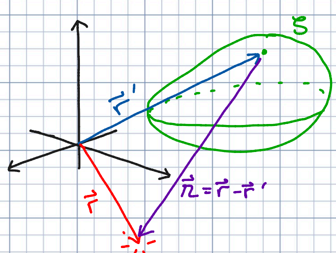
$$Q = \int_S dq(\vec{r}') = \int_S da'\, \sigma(\vec{r}')$$

Visit every point on surface S & add up the bits of charge

The amount of charge per bit of area may vary from point-to-point so it depends on <u>where</u> we are on surface.

- If the charge is spread <u>uniformly</u> over the surface then the surface charge density is the total charge divided by the area of the surface: $\sigma = Q/A$. In that case – and <u>only</u> that case – σ can be pulled out of integrals.

- The electric field produced at point $\vec{r}$ by charge spread out over a surface S is, again, just the sum of the electric fields produced by all the infinitesimal bits of charge $dq(\vec{r}')$ spread over the surface.



Visit every pt on S

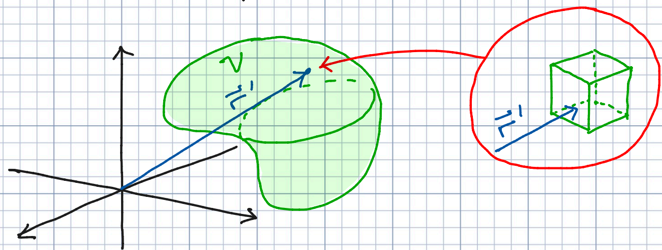
Add that point's contribution to electric field @ $\vec{r}$.

$$\vec{E}(\vec{r}) = \int_S \overbrace{\frac{1}{4\pi\varepsilon_0} dq(\vec{r}') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_S da'\, \sigma(\vec{r}') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_S da'\, \sigma(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

- And finally, we might spread a total charge Q throughout the interior of some volume $\mathcal{V}$. The (volume) charge density $\rho(\vec{r}\,')$ tells us how much charge is found in the infinitesimal volume $d\tau'$ located @ point $\vec{r}\,'$ within the volume.



The infinitesimal volume $d\tau'$ @ $\vec{r}\,'$ contains charge

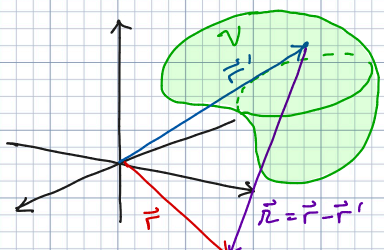
$$dq(\vec{r}\,') = d\tau'\rho(\vec{r}\,')$$

- Adding up all the bits of charge within the volume gives the total charge:

$$Q = \int_{\mathcal{V}} dq(\vec{r}\,') = \int_{\mathcal{V}} d\tau'\,\rho(\vec{r}\,')$$

- If the charge is spread <u>uniformly</u> throughout $\mathcal{V}$ then ρ is a constant equal to Q divided by the volume V: $\rho = Q/V$. But we will very often consider charge densities that vary from point to point within the volume!

- The electric field @ $\vec{r}$ is:



$$\vec{E}(\vec{r}) = \int_{\mathcal{V}} \frac{1}{4\pi\varepsilon_0}\, dq(\vec{r}\,')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} d\tau'\,\rho(\vec{r}\,')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

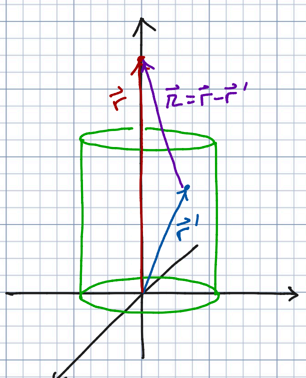
$$= \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} d\tau'\,\rho(\vec{r}\,')\, \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

- In all these cases, we set up & evaluate the integrals the same way.

  1) Set up coords that make it easy to describe the charge distribution
  2) Find $\vec{r}\,'$ for points on $\mathcal{P}$, $\mathcal{S}$, or $\mathcal{V}$
  3) Work out $\hat{\mathscr{r}}/\mathscr{r}^2$
  4) Assemble the full integrand
  5) Set up & evaluate the integral.

- Be careful! If the charge density varies from point-to-point you can't pull it outside the integral. (This is a common mistake).

- As an example, suppose I have a cylinder w/ height h & radius R, and charge density that depends on the height of the point above the base of the cylinder. What's the Coulomb integral for the Electric field @ a point on the z-axis outside the cylinder?



1) Cylinder: Use CPC w/ $0 \le s' \le R$, $0 \le z' \le h$, and $0 \le \phi' < 2\pi$. Base of cylinder is @ $z'=0$. Charge density depends on height above base, so its a function of z: $\rho(z)$. (I didn't specify what the function is!) Use CPC & Cartesian unit vectors!

2) $\vec{r}\,' = s'\hat{s} + z'\hat{z} = \underbrace{s'\cos\phi'\,\hat{x} + s'\sin\phi'\,\hat{y} + z'\hat{z}}$

$\vec{r} = 0\,\hat{x} + 0\,\hat{y} + z\,\hat{z}$

3) $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = -s'\cos\phi'\,\hat{x} - s'\sin\phi'\,\hat{y} + (z-z')\,\hat{z}$

$$|\vec{\mathscr{r}}| = \sqrt{s'^2\cos^2\phi' + s'^2\sin^2\phi' + (z-z')^2} = \sqrt{s'^2 + (z-z')^2}$$

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{-s'\cos\phi'\,\hat{x} - s'\sin\phi'\,\hat{y} + (z-z')\,\hat{z}}{\left(s'^2 + (z-z')^2\right)^{3/2}}$$

4) $d\tau' = s'\,d\phi'\,ds'\,dz'$ in CPC

$$\hookrightarrow d\tau'\,\rho(z)\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = d\phi'\,ds'\,dz'\,\rho(z')\,\frac{\left(-s'^2\cos\phi'\,\hat{x} - s'^2\sin\phi'\,\hat{y} + s'(z-z')\,\hat{z}\right)}{\left(s'^2 + (z-z')^2\right)^{3/2}}$$

5) $$\vec{E}(0,0,z) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi' \int_0^h dz' \int_0^R ds'\,\rho(z')\,\frac{\left(-s'^2\cos\phi'\,\hat{x} - s'^2\sin\phi'\,\hat{y} + s'(z-z')\,\hat{z}\right)}{\left(s'^2 + (z-z')^2\right)^{3/2}}$$

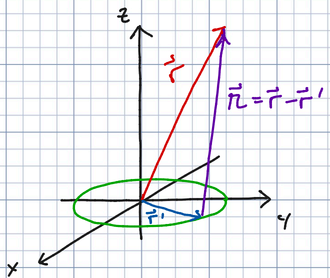
This integral gives you $\vec{E}$ @ the point $(0,0,z)$. You can't give a complete answer w/out knowing $\rho(z)$, which I did not specify. However, you <u>can</u> show that E_x & E_y are both zero @ points on the z-axis (but not @ other points). Notice that the <u>only</u> ϕ' dependence in the integrand is $\cos\phi'$ in the x-comp. and $\sin\phi'$ in the y-comp. Integrating either of these over ϕ' from 0 to 2π gives 0:

$$\int_0^{2\pi} d\phi' \cos\phi' = \int_0^{2\pi} d\phi' \sin\phi' = 0$$

$$\hookrightarrow \vec{E}(0,0,z) = \frac{1}{2\varepsilon_0}\,\hat{z}\int_0^h dz' \int_0^R ds'\,\rho(z')\,\frac{s'\times(z-z')}{\left(s'^2 + (z-z')^2\right)^{3/2}}$$

Notice that we integrate over <u>all</u> the prime coordinates, so our final answer depends only on z, h, R, & the details of ρ!

- What about a constant λ on a circular wire of radius R? What is $\vec{E}$ @ an arbitrary point (x, y, z)?



1) We'll use CPC again. Put the origin @ the center of the ring. So pts on ring have $s' = R$, $z' = 0$, and $0 \le \phi' < 2\pi$.

2) $\vec{r}\,' = R\hat{s} + 0\hat{z} = R\cos\phi'\,\hat{x} + R\sin\phi'\,\hat{y} + 0\,\hat{z}$

$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

3) $\vec{\mathscr{r}} = (x - R\cos\phi')\,\hat{x} + (y - R\sin\phi')\,\hat{y} + z\,\hat{z}$

$$|\vec{\mathscr{r}}| = \sqrt{(x - R\cos\phi')^2 + (y - R\sin\phi')^2 + z^2}$$

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{(x - R\cos\phi')\,\hat{x} + (y - R\sin\phi')\,\hat{y} + z\,\hat{z}}{\left((x - R\cos\phi')^2 + (y - R\sin\phi')^2 + z^2\right)^{3/2}}$$

4) $d\ell' = R\,d\phi'$, $\lambda = \text{const.}$

5) $$\vec{E}(x, y, z) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi' R\,\lambda\,\frac{(x - R\cos\phi')\,\hat{x} + (y - R\sin\phi')\,\hat{y} + z\,\hat{z}}{\left((x - R\cos\phi')^2 + (y - R\sin\phi')^2 + z^2\right)^{3/2}}$$

There are some things we could do to simplify this, but even so it's a tough integral. Coulomb integrals are usually very difficult unless the charge distribution is highly symmetrical <u>and</u> we try to calculate $\vec{E}$ @ a point that doesn't spoil that symmetry!

- One last note: We use primed coordinates to label points along a curve, on a surface, or in a volume that contain charge. We <u>integrate</u> over those variables when we calculate $\vec{E}$, so the final result <u>should</u> <u>not</u> contain any primed coordinates in it!

- In the last example $\vec{E}$ depends on the coordinates $\vec{r}$ of the point where we are computing the field, as well as the radius R of the ring and the constant charge-per-unit-length λ. But it <u>does</u> <u>not</u> depend on ϕ'! That refers to a specific point on the ring, and we visit <u>all</u> those points in the integral.