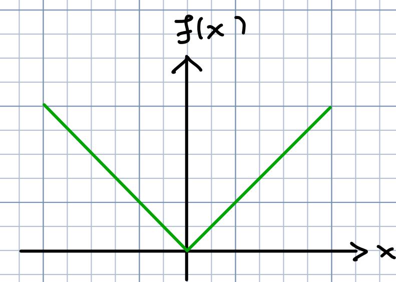
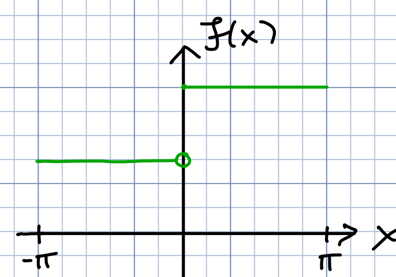
▨ ANOTHER PIECEWISE FUNCTION EXAMPLE...

- A common misconception when 1^{st} learning about Fourier Series is that you have to work out multiple series for a function w/ a piecewise definition. This is not the case! We just work out one F.S. for the whole thing.
- A function is "piecewise" if we have to break its definition up into different intervals & write it differently on each one.
- Some examples:

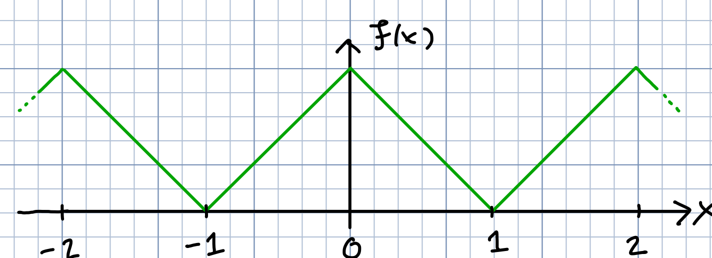
$$f(x) = \text{Abs}(x) = \begin{cases} -x, & x \le 0 \\ x, & x \geq 0 \end{cases}$$



$$f(x) = \begin{cases} 1, & -\pi \le x < 0 \\ 2, & 0 \le x \le \pi \end{cases}$$



$$f(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ f(x+2) = f(x) & \end{cases}$$



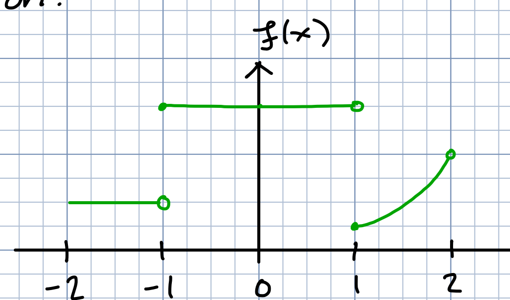
- Piecewise functions are used whenever we are working w/ a quantity that is discontinuous or has derivatives that are discontinuous

- Examples are the binary voltages used in digital circuits, where V = +5 volts for "on" & 0 volts for "off", or the periodic triangular wave in the last example.

- (In that last example I specified the behavior of $f(x)$ outside of $[-1,1]$ by stating that $f(x+2) = f(x)$. So any x can be followed back to a value in $[-1,1]$. For instance, $f(39.5) = f(37.5) = \ldots = f(1.5) = f(0.5) = 0.5$.)

- When we evaluate integrals involving a function w/ a piecewise definition we have to break it up into sub-integrals according to the definition of the function.

$$f(x) = \begin{cases} 1, & -2 \le x < -1 \\ 3, & -1 \le x < 1 \\ \frac{1}{2}x^2, & 1 \le x \le 2 \end{cases}$$



$$\int_{-2}^{2} dx\, f(x) = \int_{-2}^{-1} dx\, 1 + \int_{-1}^{1} dx\, 3 + \int_{1}^{2} dx\, \frac{1}{2}x^2$$

$$= x\Big|_{-2}^{-1} + 3x\Big|_{-1}^{1} + \frac{1}{6}x^3\Big|_{1}^{2}$$

$$= -1-(-2) + 3-(-3) + \frac{1}{6}\cdot 2^3 - \frac{1}{6}\cdot 1^3$$

$$= 1 + 6 + \frac{7}{6} = \frac{49}{6}$$

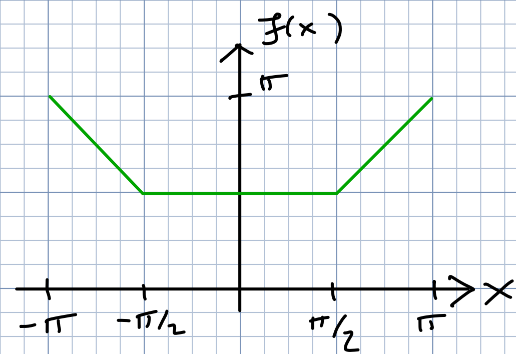
If $x_1 < x_2 < x_3 < \ldots < x_n$ Then

$$\int_{x_1}^{x_n} dx\, f(x) = \int_{x_1}^{x_2} dx\, f(x) + \int_{x_2}^{x_3} dx\, f(x) + \ldots + \int_{x_{n-1}}^{x_n} dx\, f(x)$$

- When we work out the F.S. of a piecewise function we'll need to do this for the integrals we evaluate to get a_0, a_n, & b_n.

- **EXAMPLE:**

$$f(x) = \begin{cases} -x \,, & -\pi \le x \le \pi/2 \\ \pi/2 \,, & -\pi/2 \le x \le \pi/2 \\ x \,, & \pi/2 \le x \le \pi \end{cases}$$



Even on $[-\pi, \pi]$, so expect $b_n = 0$!

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, f(x)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} dx\,(-x) + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx\,\frac{\pi}{2} + \frac{1}{2\pi}\int_{\pi/2}^{\pi} dx\, x$$

$$= \frac{1}{2\pi}\left(-\frac{1}{2}x^2\Big|_{-\pi}^{-\pi/2}\right) + \frac{1}{2\pi}\left(\frac{\pi}{2}x\Big|_{-\pi/2}^{\pi/2}\right) + \frac{1}{2\pi}\cdot\left(\frac{1}{2}x^2\Big|_{\pi/2}^{\pi}\right)$$

$$= \frac{1}{2\pi}\left[-\frac{1}{2}\cdot\frac{\pi^2}{4} + \frac{1}{2}\cdot\pi^2 + \frac{\pi^2}{4} - \left(-\frac{\pi^2}{4}\right) + \frac{1}{2}\pi^2 - \frac{1}{2}\cdot\frac{\pi^2}{4}\right]$$

$$\Rightarrow a_0 = \frac{5\pi}{8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x)\cos(nx) = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} dx\,(-x)\cos(nx) + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx\cdot\frac{\pi}{2}\cdot\cos(nx)$$

$$+ \frac{1}{\pi}\int_{\pi/2}^{\pi} dx\cdot x\cdot\cos(nx)$$

Use I.B.P. $\int dx\, x\cos(nx) = \frac{x\sin(nx)}{n} + \frac{1}{n^2}\cos(nx)$

$$= \frac{1}{\pi}\left(-\frac{x}{n}\sin(nx) - \frac{1}{n^2}\cos(nx)\right)\Big|_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left(\frac{\pi}{2}\frac{1}{n}\sin(nx)\Big|_{-\pi/2}^{\pi/2}\right)$$

$$+ \frac{1}{\pi}\left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)\Big|_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\sin\left(-n\frac{\pi}{2}\right) - \frac{1}{n^2}\cos(n\pi/2) - \frac{\pi}{n}\cancelto{0}{\sin(n\pi)} + \frac{1}{n^2}\cos(n\pi) + \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right)\right.$$

$$\left. - \frac{\pi}{2n}\sin\left(-\frac{n\pi}{2}\right) + \frac{\pi}{n}\cancelto{0}{\sin(n\pi)} + \frac{1}{n^2}\cos(n\pi) - \frac{\pi}{2n}\sin\left(n\frac{\pi}{2}\right) - \frac{1}{n^2}\cos\left(n\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n} \overbrace{\sin\left(-n\frac{\pi}{2}\right)}^{-\sin\left(\frac{n\pi}{2}\right)} - \frac{1}{n^2}\cos\left(n\pi/2\right) - \frac{\pi}{n}\cancel{\sin(n\pi)}^{\,0} + \frac{1}{n^2}\cos(n\pi) + \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) \right.$$

$$\left. - \frac{\pi}{2n}\overbrace{\sin\left(-\frac{n\pi}{2}\right)}^{-\sin(n\pi/2)} + \frac{\pi}{n}\cancel{\sin(n\pi)}^{\,0} + \frac{1}{n^2}\cos(n\pi) - \frac{\pi}{2n}\sin\left(n\frac{\pi}{2}\right) - \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right]$$

$$a_n = \frac{1}{\pi}\left[\cancel{-\frac{\pi}{n}\sin\left(\frac{n\pi}{2}\right)} - \frac{2}{n^2}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n^2}\cos(n\pi) \cancel{+\frac{\pi}{n}\sin\left(\frac{n\pi}{2}\right)} \right]$$

This is a perfectly good result for a_n, though it's a little messy. Can we simplify it @ all? Usually when we have trig functions evaluated @ integer multiples of π or $\pi/2$ we can write the result in terms of $(-1)^n$ or $1 \pm (-1)^n$, etc.

$$a_n = \frac{1}{\pi}\left[-\frac{2}{n^2}\underbrace{\cos\left(\frac{n\pi}{2}\right)}_{\substack{0 \text{ if } n \text{ odd} \\ (-1)^{n/2} \text{ if } n \text{ even}}} + \frac{2}{n^2}\underbrace{\cos(n\pi)}_{(-1)^n} \right]$$

One way to write this is to consider even & odd values of n separately.

EVEN: $n = 2,4,6,\ldots \rightarrow n = 2k$ w/ $k = 1,2,3,\ldots$

$$a_{2k} = \frac{1}{\pi}\left[-\frac{\cancel{2}}{\cancel{4}_{2}k^2}\overbrace{\cos(k\pi)}^{(-1)^k} + \frac{\cancel{2}}{\cancel{4}_{2}k^2}\underbrace{\cos(2k\pi)}_{(-1)^{2k} = ((-1)^2)^k = 1} \right]$$

$$\Rightarrow a_{2k} = \frac{1}{2\pi k^2} \times \left(1 - (-1)^k\right)$$

ODD: $n = 1,3,5,7,\ldots \rightarrow n = 2k+1$ w/ $k = 0,1,2,\ldots$

$$a_{2k+1} = \frac{1}{\pi}\left[-\frac{2}{(2k+1)^2}\overbrace{\cos\left(\left(k+\frac{1}{2}\right)\pi\right)}^{0} + \frac{2}{(2k+1)^2}\underbrace{\cos\left((2k+1)\pi\right)}_{-1} \right]$$

$$\Rightarrow a_{2k+1} = -\frac{2}{\pi(2k+1)^2}$$

Now for the b_n, which should be zero.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x)\sin(nx)$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} dx\,(-x)\sin(nx) + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx\,\frac{\pi}{2}\sin(nx) + \frac{1}{\pi}\int_{\pi/2}^{\pi} dx\, x\sin(nx)$$

First let's show $b_n = 0$ by directly evaluating the integrals:

$$= \frac{1}{\pi}\left(\frac{x}{n}\cos(nx) - \frac{1}{n^2}\sin(nx)\Big|_{-\pi}^{-\pi/2}\right) + \frac{1}{\pi}\left(-\frac{\pi}{2n}\cos(nx)\Big|_{-\pi/2}^{\pi/2}\right)$$

$$+ \frac{1}{\pi}\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Big|_{\pi/2}^{\pi}$$

cos EVEN

sin ODD

$$= \frac{1}{\pi}\Bigg[\cancel{-\frac{\pi}{2n}\overbrace{\cos\left(-\frac{\pi n}{2}\right)}^{\cos\left(\frac{\pi n}{2}\right)}} \cancel{- \frac{1}{n^2}\overbrace{\sin\left(-\frac{n\pi}{2}\right)}^{-\sin(n\pi/2)}} \cancel{+ \frac{\pi}{n}\overbrace{\cos(-n\pi)}^{\cos(n\pi)}} \cancel{+ \frac{1}{n^2}\overbrace{\sin(-n\pi)}^{-\sin(n\pi)}}$$

$$\cancel{-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)} \cancel{+ \frac{\pi}{2n}\overbrace{\cos\left(-\frac{n\pi}{2}\right)}^{\cos(n\pi/2)}} \cancel{- \frac{\pi}{n}\cos(n\pi)} \cancel{+ \frac{1}{n^2}\sin(n\pi)}$$

$$\cancel{+\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)} \cancel{- \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)}\Bigg]$$

$$\Rightarrow b_n = 0$$

Alternately, you could use $f(-x) = f(x)$ on $[-\pi, \pi]$ while $\sin(-nx) = -\sin(nx)$ to show this <u>before</u> evaluating the integral.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} dx\, f(x)\sin(nx) + \frac{1}{\pi}\int_{0}^{\pi} dx\, f(x)\sin(nx)$$

$u = -x \quad dx = -du$

$$= \frac{1}{\pi}\int_{\pi}^{0}(-du)\,\underbrace{f(-u)}_{f(u)}\,\underbrace{\sin(-nu)}_{-\sin(nu)} + \frac{1}{\pi}\int_0^{\pi} dx\, f(x)\sin(nx)$$

$$\int_A^B = -\int_B^A$$

$$= -\frac{1}{\pi}\int_0^{\pi} du\, f(u)\sin(nu) + \frac{1}{\pi}\int_0^{\pi} dx\, f(x)\sin(nx) = 0$$

So our Fourier Series for $f(x)$ is:

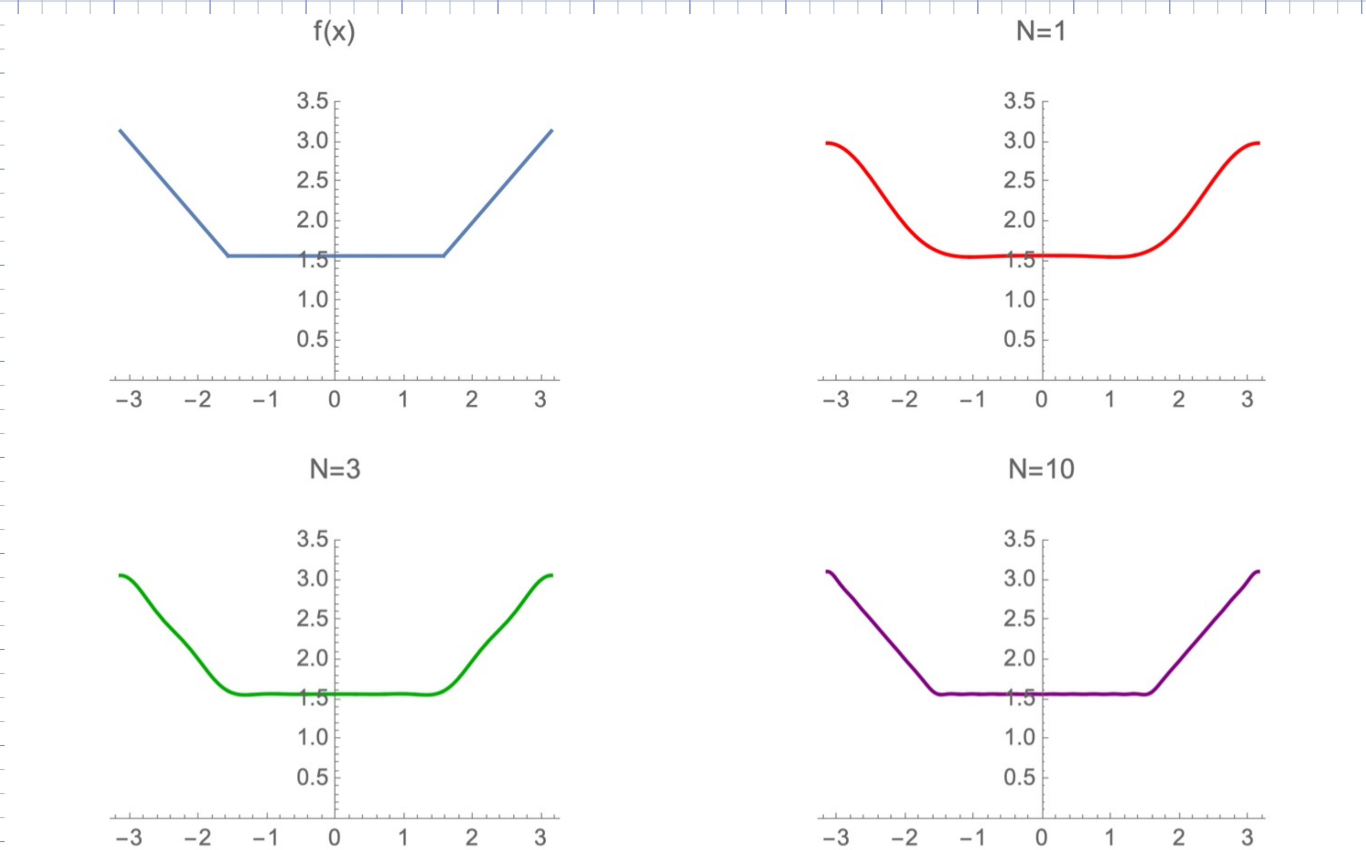
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} \cancel{b_n}^{\,0} \sin(nx)$$

$$= a_0 + \underbrace{\sum_{k=1}^{\infty} a_{2k} \cos(2kx)}_{\text{even } n=2k} + \underbrace{\sum_{k=0}^{\infty} a_{2k+1} \cos((2k+1)x)}_{\text{odd } n=2k+1}$$

$$f(x) = \frac{5\pi}{8} + \sum_{k=1}^{\infty} \frac{1-(-1)^k}{2\pi k^2} \cos(2kx) + \sum_{k=0}^{\infty} \frac{-2}{\pi(2k+1)^2} \cos((2k+1)x)$$

Did we get it right? Let's plot the first N terms in each sum & compare w/ our original function. That is, we'll plot $f(x)$ and

$$\frac{5\pi}{8} + \sum_{k=1}^{N} \frac{1-(-1)^k}{2\pi k^2} \cos(2kx) + \sum_{k=0}^{N} \frac{-2}{\pi(2k+1)^2} \cos((2k+1)x)$$



- Now here's the important point: We only worked out <u>one</u> F.S. & it describes the whole piecewise function everywhere on $[-\pi, \pi]$. We did not write a separate F.S. for the 3 intervals $[-\pi, -\pi/2]$, $[-\pi/2, \pi/2]$, and $[\pi/2, \pi]$ where $f(x)$ has distinct definitions!